CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #10

Problem 10.1

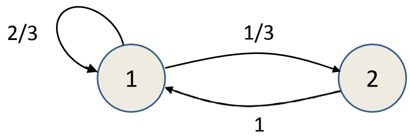
Determine whether or not the following matrices could be a transition matrix for a Markov chain. For those that are not, explain why not, and for those that are, draw a picture of the chain.

(a) $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ (b) $\begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 0.5 \end{bmatrix}$ (d) $\begin{bmatrix} 1/4 & 2 & 1 \\ 3/4 & 0 & 2/3 \\ 0 & 1 & 3 \end{bmatrix}$

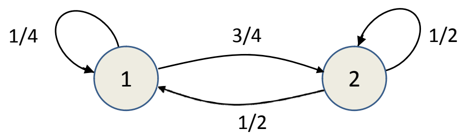
Answer

(a) This is not a valid transition matrix because the columns do not sum to one.

(b) This is a valid transition matrix, and a diagram of the chain is given below,



(c) This is a valid transition matrix, and a diagram of the chain is given below,



(d) This is not a valid transition matrix because some of the elements are greater than one.

Problem 10.2

For a Markov chain characterized by the following transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 & 0.4 \end{bmatrix}$$

Find the probability that state 2 changes to state 4 on the sixth step.

Answer

$$\mathbf{P}^6 = \begin{bmatrix} 0.1720 & 0.1718 & 0.1723 & 0.1718 & 0.1717 \\ 0.1764 & 0.1767 & 0.1760 & 0.1766 & 0.1768 \\ 0.1675 & 0.1672 & 0.1683 & 0.1673 & 0.1670 \\ 0.1876 & 0.1874 & 0.1880 & 0.1875 & 0.1873 \\ 0.2965 & 0.2969 & 0.2955 & 0.2967 & 0.2971 \end{bmatrix}$$

so the probability that state 2 changes to state 4 is 0.1874.

Problem 10.3

Which of the following transition matrices are regular?

(a) $\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0.6 \\ 0 & 0.4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0.4 & 1 \\ 1 & 0.2 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 1 & 0.5 \\ 0.5 & 0 & 0.1 \\ 0.2 & 0 & 0.4 \end{bmatrix}$

Answer ____

Transition matrices (a), (c), and (d) are regular.

Problem 10.4

Find the steady state vectors of each of the following regular transition matrices:

(a) $\begin{bmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{4} \end{bmatrix}$ (b) $\begin{bmatrix} 0.81 & 0.26 \\ 0.19 & 0.74 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

Answer ____

(a) $\mathbf{v} = [9/17,\ 8/17]^T$
(b) $\mathbf{v} = [26/45,\ 19/45]^T$
(c) $\mathbf{v} = [3/19,\ 4/19,\ 12/19]^T$

Problem 10.5

(a) If $\mathbf{P}$ is a regular $n \times n$ stochastic matrix with steady state vector $\mathbf{q}$, and if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors, what can you say about the behavior of the sequence

$$\mathbf{P}\mathbf{e}_i, \quad \mathbf{P}^2\mathbf{e}_i, \quad \mathbf{P}^3\mathbf{e}_i, \ldots, \quad \mathbf{P}^k\mathbf{e}_i, \ldots$$

as $k \to \infty$?

(b) What does this tell you about the behavior of the column vectors of $\mathbf{P}^k$ as $k \to \infty$?

Solution ____

(a) If $\mathbf{P}$ is a regular stochastic matrix, we know that for **any** initial vector $\mathbf{x}(0)$, the sequence of vectors $\mathbf{x}(n) = \mathbf{P}^n\mathbf{x}(0)$ converges to a fixed vector $\mathbf{v}$ as n approaches infinity, and that for large n

$$\mathbf{P}^n\mathbf{x}(0) \approx \mathbf{v}$$

Since the vector $\mathbf{x}(0)$ is arbitrary, we can let $\mathbf{x}(0)$ be any of the standard unit vectors, $\mathbf{x}(0) = \mathbf{e}_i$. Therefore, for large n

$$\mathbf{P}^n\mathbf{e}_i \approx \mathbf{v}$$

(b) Since $\mathbf{P}^n\mathbf{e}_i$ is the ith column of $\mathbf{P}^n$, and since for large n

$$\mathbf{P}^n\mathbf{e}_i \approx \mathbf{v}$$

then it follows that

$$\lim_{n\to\infty} \mathbf{P}^n = [\mathbf{v},\ \mathbf{v},\ \ldots,\ \mathbf{v}]$$

Problem 10.6★

A transition matrix has the property that all of its elements are between zero and one, and the sum of the elements in each column is equal to one. Suppose that, in addition, the sum of the elements along each row is equal to one, such as the matrix given below.

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

In this case, the transpose of the transition matrix is also a transition matrix.

(a) Show that if a $k \times k$ transition matrix is regular and all of its rows sum to one then the elements of the steady-state vector are all equal to $1/k$.

(b) Show that the matrix $\mathbf{P}$ above is regular and find its steady-state vector.

Solution

(a) The steady state vector is the solution to the eigenvalue equation

$$\mathbf{P}\mathbf{v} = \mathbf{v}$$

Therefore,

$$\sum_{i=1}^{k} p_{ji}v_i = v_j \quad ; \quad j = 1, 2, \ldots k$$

If $v_i = 1/k$ for $i = 1, 2, \ldots, k$ then

$$\sum_{i=1}^{k} p_{ji}v_i = \frac{1}{k}\sum_{i=1}^{k} p_{ji} = \frac{1}{k} \quad ; \quad j = 1, 2, \ldots k$$

Since all of the rows of $\mathbf{P}$ sum to one, then the sum above is equal to one and it follows that

$$\mathbf{v} = \frac{1}{k}[1,\ 1,\ \ldots, 1]^T$$

is a solution to the eigenvector equation $\mathbf{P}\mathbf{v} = \mathbf{v}$ and, therefore, must be the steady state vector.

(b) Since all of then entries in $\mathbf{P}^2$ are greater than zero, then $\mathbf{P}$ is a regular matrix. Therefore, it follows that the steady state vector is

$$\mathbf{v} = [1/3,\ 1/3,\ 1/3]^T$$

Problem 10.7★

Find all absorbing states for the transition matrices given below.

(a) $\begin{bmatrix} 0.15 & 0 & 0.4 \\ 0.05 & 1 & 0.6 \\ 0.8 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0.4 & 0 & 0.9 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.1 \end{bmatrix}$ (c) $\begin{bmatrix} 0.2 & 0 & 0.9 & 0 \\ 0.5 & 1 & 0.02 & 0 \\ 0.1 & 0 & 0.04 & 0 \\ 0.2 & 0 & 0.04 & 1 \end{bmatrix}$

Solution

(a) State 2 is an absorbing state.

(b) State 2 is an absorbing state.

(c) States 2 and 4 are absorbing states.

Problem 10.8★

Find the fundamental matrix for the absorbing Markov chains with the given transition matrix, and also find the product matrix **RF**.

(a) $\begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 1 & 0.3 \\ 0 & 0 & 0.5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1/3 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 2/3 & 1 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$ (d) $\begin{bmatrix} 0.4 & 0 & 0 & 0.1 & 0 \\ 0.2 & 1 & 0 & 0.5 & 0 \\ 0.3 & 0 & 1 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0.1 & 0 & 0 & 0.2 & 1 \end{bmatrix}$

Solution

(a) The transition matrix

$$\left[\begin{array}{cc:c} 1 & 0 & 0.2 \\ 0 & 1 & 0.3 \\ \hdashline 0 & 0 & 0.5 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I}_2 & \mathbf{r} \\ \hdashline \mathbf{0}^T & q \end{array}\right]$$

has the desired form with the first two states being absorbing, so $q = 0.5$ and the fundamental matrix is

$$f = (1-q)^{-1} = 2$$

and

$$\mathbf{r}f = 2[0.2,\ 0.3]^T = [0.4,\ 0.6]^T$$

(b) The transition matrix

$$\left[\begin{array}{cc:c} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ \hdashline 0 & 0 & 1/3 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I}_2 & \mathbf{r} \\ \hdashline \mathbf{0}^T & q \end{array}\right]$$

is again in the proper form with the first two states being absorbing. So $q = 1/3$, the fundamental matrix is

$$f = (1-q)^{-1} = 3/2$$

and

$$\mathbf{r}f = \frac{3}{2}[1/3,\ 1/3]^T = [1/2,\ 1/2]^T$$

(c) The transition matrix

$$\begin{bmatrix} 1 & 1/3 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 2/3 & 1 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

has two absorbing states, states 1 and 3. Therefore, to find the fundamental matrix, it is necessary to renumber the states by changing states 2 and 3. This is done by swapping rows 2 and 3 and columns and 3, which leads to the following transition matrix,

$$\left[\begin{array}{cc:cc} 1 & 0 & 1/3 & 1/4 \\ 0 & 1 & 2/3 & 1/4 \\ \hdashline 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I}_2 & \mathbf{R} \\ \hdashline \mathbf{0} & \mathbf{Q} \end{array}\right]$$

Therefore

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} 1 & -1/4 \\ 03/4 & \end{bmatrix}^{-1} = \frac{4}{3}\begin{bmatrix} 3/4 & 1/4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix}$$

and

$$\mathbf{RF} = \begin{bmatrix} 1/3 & 1/4 \\ 2/3 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 1/3 \\ 0 & 4/3 \end{bmatrix} = \begin{bmatrix} 1/3 & 4/9 \\ 2/3 & 5/9 \end{bmatrix}$$

(d) The transition matrix

$$\begin{bmatrix} 0.4 & 0 & 0 & 0.1 & 0 \\ 0.2 & 1 & 0 & 0.5 & 0 \\ 0.3 & 0 & 1 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0.1 & 0 & 0 & 0.2 & 1 \end{bmatrix}$$

represents a Markov chain with three absorbing states, state 2, 3, and 4. If we renumber the states, swapping states 1 and 4 by interchanging rows 1 and 4 and columns 1 and 4 we have the transition matrix

$$\left[\begin{array}{ccc:cc} 1 & 0 & 0 & 0.2 & 0.1 \\ 0 & 1 & 0 & 0.5 & 0.2 \\ 0 & 0 & 1 & 0.1 & 0.3 \\ \hdashline 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0.4 \end{array}\right] = \left[\begin{array}{c:c} \mathbf{I}_3 & \mathbf{R} \\ \hdashline \mathbf{0} & \mathbf{Q} \end{array}\right]$$

Therefore,

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} 0.9 & 0 \\ -0.1 & 0.6 \end{bmatrix}^{-1} = \frac{1}{27}\begin{bmatrix} 30 & 0 \\ 5 & 45 \end{bmatrix}$$

and

$$\mathbf{RF} = \begin{bmatrix} 0.2 & 0.1 \\ 0.5 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \frac{1}{27}\begin{bmatrix} 30 & 0 \\ 5 & 45 \end{bmatrix} = \begin{bmatrix} 13/54 & 1/6 \\ 16/27 & 1/3 \\ 1/6 & 1/2 \end{bmatrix}$$

Problem 10.9★

Physical traits are determined by the genes that an offspring receives from its parents. In the simplest case, a trait in the offspring is determined by one pair of genes, one member of the pair is inherited from the male parent and the other from the female parent. Typically, each gene in a pair can assume one of two forms, called *alleles*, denoted by A and a. This leads to three possible pairings,

$$AA, \qquad Aa, \qquad aa$$

called *genotypes* (the pairs Aa and aA result in the same trait and are therefore indistinguishable from the each other). It is shown in the study of heredity that if a parent of known genotype is crossed with a random parent of unknown genotype, then the offspring will have the genotype probabilities given in the following table, which can be viewed as a transition matrix for a Markov process:

		Genotype of Parent		
		AA	**Aa**	**aa**
Genotype of Offspring	**AA**	$\frac{1}{2}$	$\frac{1}{4}$	0
	Aa	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	aa	0	$\frac{1}{4}$	$\frac{1}{2}$

Thus, for example, the offspring of a parent of genotype AA that is crossed at random with a parent of unknown genotype will have a 50% chance of being AA, a 50% chance of being Aa and no chance of being aa.

(a) Show that this transition matrix is regular.

(b) Find the steady state vector, and discuss what it means physically.

Solution

(a) The transition matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Since

$$\mathbf{p}^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} & \frac{3}{2} \end{bmatrix}$$

then $\mathbf{P}$ is a regular matrix.

(b) To find the steady-state vector, we need to solve the following equation for $\mathbf{v}$

$$(\mathbf{P} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

With

$$\mathbf{P} - \mathbf{I} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

finding the row echelon form we have

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, v_3 is a free variable, and $-\frac{1}{4}v_2 + \frac{1}{2}x_3 = 0$ and $-\frac{1}{1}v_1 + \frac{1}{4}x_2 = 0$. Thus, we have

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

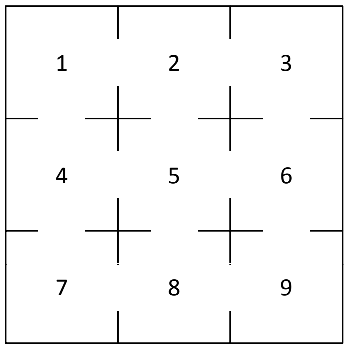
Note, however, that we need to scale $\mathbf{v}$ so that it is a valid probability vector. Specifically, we want the sum of the elements to equal one. Dividing each element by 4 we have the desired result,

$$\mathbf{v} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

What this steady-state vector says is that eventually half of the population will be of type Aa and one fourth of the population will be either AA or aa. What this is equivalent to is the following random experiment. Suppose that we have two coins, each one marked with an A on one side and with an a on the other. We flip each coin once, and note what side is showing. There is only one way to get two As - each coin showing an A. Similarly, there is only one way to get two as - each coin shown an a. However, there are two ways to get one A and one a - coin 1 with an A and coin two with an a, and coin 1 with an a and coin two with an A. One out of four possibilities for either AA or aa means a probability of 0.25 and two chances out of four for Aa means a probability of 0.5.

Problem 10.10*

A mouse is placed in a box with nine rooms as illustrated in the figure below.



Assume that, at regular intervals of time, it is equally likely that the mouse will decide to go through any door in the room or stay in the room.

(a) Construct the 9×9 transition matrix for this problem and show that it is regular.

(b) Determine the steady-state vector for this matrix.

(c) Use a symmetry argument to show that this problem may be solved using only a 3×3 matrix.

Solution

(a) In this problem, we have nine states, each state corresponding to a room in a 3×3 arrow of rooms and hallways. The mouse may move from one room to another or remain in the same room, each with the same probability of occurrence. Therefore, we can assign transition probabilities in moving from one state (room) to another as illustrated in the following table,

	1	2	3	4	5	6	7	8	9
1	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0
2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	0	$\frac{1}{5}$	0	0	0	0
3	0	$\frac{1}{4}$	$\frac{1}{3}$	0	0	$\frac{1}{4}$	0	0	0
4	$\frac{1}{3}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$	0	$\frac{1}{3}$	0	0
5	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0
6	0	0	$\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{4}$	0	0	$\frac{1}{3}$
7	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{3}$	$\frac{1}{4}$	0
8	0	0	0	0	$\frac{1}{5}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$
9	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{3}$

For example, when the mouse is in room 1, he will either stay in room one or move to rooms two or four, all with equal probability of $\frac{1}{3}$. Similarly, if the mouse is in room 2, he will either stay there or move to rooms 1, 3, or 5, all with equal probability of $\frac{1}{4}$. From this table, we immediately have the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{3} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{4} & \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{4} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

which is easily seen to be a valid stochastic matrix since all entries are between zero and one and all columns sum to one.

We can see that $\mathbf{P}$ is regular by looking at $\mathbf{P}^n$. Although $\mathbf{P}$ contains many zeros, all of the entries in $\mathbf{P}^4$ are non-zero.

(b) Since $\mathbf{P}$ is regular, $\mathbf{x}(n)$ will converge to a steady-state vector $\mathbf{v}$. We may find $\mathbf{v}$ by either forming the matrix $\mathbf{P}^n$ for a sufficiently large value of n, one for which $\mathbf{P}^{n+1} \approx \mathbf{P}^n$, and then $\mathbf{v}$ will be approximately equal to any one of the columns of $\mathbf{P}^n$. This is true because $\mathbf{v}$ is independent of the starting state, $\mathbf{x}(0)$. Therefore,

$$\mathbf{v} \approx \mathbf{P}^n \mathbf{e}_i$$

where $\mathbf{e}_i$ is a unit vector with all elements equal to zero except the ith element, which is equal to one, and $\mathbf{P}^n \mathbf{e}_i$ is the ith column of $\mathbf{P}^n$.

Alternatively, we can find the eigenvector corresponding to an eigenvalue of one,

$$(\mathbf{P} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

Here we will take this approach. After entering the values of the matrix $\mathbf{P}$ into MATLAB or Octave, we use the `eig` m-file as follows,

```
>> [v,e]=eig(P);
```

We then look at the vector in `v` corresponding to the eigenvalue of one. In my case it is the first column of the matrix `v` (it may be a different one for you), and we find

```
>> v(:,1)
ans =
     -0.26833
     -0.35777
     -0.26833
     -0.35777
     -0.44721
     -0.35777
     -0.26833
     -0.35777
     -0.26833
```

Note that we need to scale $\mathbf{v}$ so that it is a valid probability vector. Specifically, we want the sum of the elements to equal one. Dividing each element by the sum of the terms in the vector we get the desired solution, which is

```
>> v(:,1)/sum(v(:,1))
ans =
      0.090909
      0.121212
      0.090909
      0.121212
```

```
0.151515
0.121212
0.090909
0.121212
0.090909
```

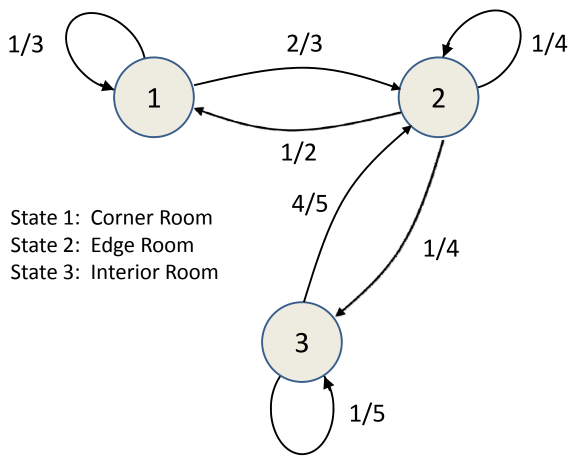
Each of the elements in $\mathbf{v}$ can be converted to a rational number. What we have for our final answer is

$$\mathbf{v} = \begin{bmatrix} 1/11 \\ 4/33 \\ 1/11 \\ 4/33 \\ 5/33 \\ 4/33 \\ 1/11 \\ 4/33 \\ 1/11 \end{bmatrix}$$

(c) If we look at the structure of the box that the mouse is moving around in, there are three types of rooms,

1. A *corner room* within with the mouse has only two options: Stay in the corner room or move to one of two possible *edge rooms*.
2. An *edge room* within which the mouse has three options: Stay in the edge room, move into one of the two neighboring edge rooms, or move into an *interior room*.
3. An *interior room* within which the mouse has two options: Stay in the *interior room* or move into one of the four possible *edge rooms*.

A state diagram of this simplified three-state Markov chain is given in the figure below.



Observe that the probabilities are assigned based on how many options there are in moving from one state to another or remaining in the same state. For example, in state 2, there are two ways to move a corner room, one way to move to an interior room, and one way to stay in the same room. Thus, the probability of moving to state 1 is $2/4$, the probability of moving to state 3 is $1/4$ and the probability of staying in the same state is $1/4$.

The transition matrix for this Markov chain is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{2}{3} & \frac{1}{4} & \frac{4}{5} \\ 0 & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

This is easily seen to be a regular matrix since $\mathbf{P}^2$ has all positive entries. The steady-state solution is

```
>> P=[1/3 1/2 0 ; 2/3 1/4 4/5 ; 0 1/4 1/5]
>> [v,e]=eig(P);
v =
   -0.58209   0.76200   0.50508
   -0.77611  -0.12700  -0.80812
   -0.24254  -0.63500   0.30305

e =

Diagonal Matrix

   1.00000         0         0
   0         0.25000         0
   0               0  -0.46667
```

Therefore, the eigenvector corresponding to an eigenvalue of one is the first column of `v`. Scaling this column vector so that the sum of the elements is equal to one we have

```
>> v(:,1)/sum(v(:,1))
ans =
      0.36364
      0.48485
      0.15152
```

What this says is that

1. The probability of being in a corner room (state 1) is 0.36364,
2. The probability of being in an edge room (state 2) is 0.48485, and
3. The probability of being in an interior room (state 3) is 0.15152.

Since there are four corner rooms, the probability of being in a *specific* corner room is $0.36364/4 = 0.090909 = 1/11$. Since there are four edge rooms, the probability of being in a *specific* edge room is $0.48485/4 = 0.121212 = 4/33$. And since there is only one interior room, then the probability of being in the interior room remains $0.151515 = 5/33$. These results, of course, are consistent with what we found for the full nine-state model for the problem.