Chung-Ang University

Linear Algebra Spring 2014

Problem Set #10

Assigned: June 04, 2014
Due Date: None

Problem 10.1

Determine whether or not the following matrices could be a transition matrix for a Markov chain. For those that are not, explain why not, and for those that are, draw a picture of the chain.

(a) $\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ (b) $\begin{bmatrix} 2/3 & 1 \\ 1/3 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 0.5 \end{bmatrix}$ (d) $\begin{bmatrix} 1/4 & 2 & 1 \\ 3/4 & 0 & 2/3 \\ 0 & 1 & 3 \end{bmatrix}$

Problem 10.2

For a Markov chain characterized by the following transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 & 0.3 & 0.4 \end{bmatrix}$$

Find the probability that state 2 changes to state 4 on the sixth step.

Problem 10.3

Which of the following transition matrices are regular?

(a) $\begin{bmatrix} 0.2 & 0.9 \\ 0.8 & 0.1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0.6 \\ 0 & 0.4 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0.4 & 1 \\ 1 & 0.2 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0.3 & 1 & 0.5 \\ 0.5 & 0 & 0.1 \\ 0.2 & 0 & 0.4 \end{bmatrix}$

Problem 10.4

Find the steady state vectors of each of the following regular transition matrices:

(a) $\begin{bmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{4} \end{bmatrix}$ (b) $\begin{bmatrix} 0.81 & 0.26 \\ 0.19 & 0.74 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$

Problem 10.5

(a) If $\mathbf{P}$ is a regular $n \times n$ stochastic matrix with steady state vector $\mathbf{q}$, and if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors, what can you say about the behavior of the sequence

$$\mathbf{P}\mathbf{e}_i, \quad \mathbf{P}^2\mathbf{e}_i, \quad \mathbf{P}^3\mathbf{e}_i, \ldots, \quad \mathbf{P}^k\mathbf{e}_i, \ldots$$

as $k \to \infty$?

(b) What does this tell you about the behavior of the column vectors of $\mathbf{P}^k$ as $k \to \infty$?

Problem 10.6

A transition matrix has the property that all of its elements are between zero and one, and the sum of the elements in each column is equal to one. Suppose that, in addition, the sum of the elements along each row is equal to one, such as the matrix given below.

$$\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

In this case, the transpose of the transition matrix is also a transition matrix.

(a) Show that if a $k \times k$ transition matrix is regular and all of its rows sum to one then the elements of the steady-state vector are all equal to $1/k$.

(b) Show that the matrix $\mathbf{P}$ above is regular and find its steady-state vector.

Problem 10.7

Find all absorbing states for the transition matrices given below.

(a) $\begin{bmatrix} 0.15 & 0 & 0.4 \\ 0.05 & 1 & 0.6 \\ 0.8 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0.4 & 0 & 0.9 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.1 \end{bmatrix}$ (c) $\begin{bmatrix} 0.2 & 0 & 0.9 & 0 \\ 0.5 & 1 & 0.02 & 0 \\ 0.1 & 0 & 0.04 & 0 \\ 0.2 & 0 & 0.04 & 1 \end{bmatrix}$

Problem 10.8

Find the fundamental matrix for the absorbing Markov chains with the given transition matrix, and also find the product matrix $\mathbf{RF}$.

(a) $\begin{bmatrix} 1 & 0 & 0.2 \\ 0 & 1 & 0.3 \\ 0 & 0 & 0.5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1/3 & 0 & 1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 2/3 & 1 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$ (d) $\begin{bmatrix} 0.4 & 0 & 0 & 0.1 & 0 \\ 0.2 & 1 & 0 & 0.5 & 0 \\ 0.3 & 0 & 1 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0.1 & 0 & 0 & 0.2 & 1 \end{bmatrix}$

Problem 10.9

Physical traits are determined by the genes that an offspring receives from its parents. In the simplest case, a trait in the offspring is determined by one pair of genes, one member of the pair is inherited from the male parent and the other from the female parent. Typically, each gene in a pair can assume one of two forms, called *alleles*, denoted by A and a. This leads to three possible pairings,

$$AA, \qquad Aa, \qquad aa$$

called *genotypes* (the pairs Aa and aA result in the same trait and are therefore indistinguishable from the each other). It is shown in the study of heredity that if a parent of known genotype is crossed with a random parent of unknown genotype, then the offspring will have the genotype probabilities given in the following table, which can be viewed as a transition matrix for a Markov process:

		Genotype of Parent		
		AA	**Aa**	**aa**
	AA	$\frac{1}{2}$	$\frac{1}{4}$	0
Genotype of Offspring	**Aa**	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	aa	0	$\frac{1}{4}$	$\frac{1}{2}$

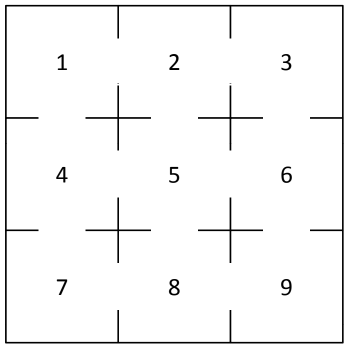
Thus, for example, the offspring of a parent of genotype AA that is crossed at random with a parent of unknown genotype will have a 50% chance of being AA, a 50% chance of being Aa and no chance of being aa.

(a) Show that this transition matrix is regular.

(b) Find the steady state vector, and discuss what it means physically.

Problem 10.10

A mouse is placed in a box with nine rooms as illustrated in the figure below.



Assume that, at regular intervals of time, it is equally likely that the mouse will decide to go through any door in the room or stay in the room.

(a) Construct the 9×9 transition matrix for this problem and show that it is regular.

(b) Determine the steady-state vector for this matrix.

(c) Use a symmetry argument to show that this problem may be solved using only a 3×3 matrix.