CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #9

Answers to Practice Problems

Problem 9.1

Suppose that $\mathbf{v}$ is an eigenvector of an $n \times n$ matrix $\mathbf{A}$, and let λ be the corresponding eigenvalue. Is $\mathbf{v}$ an eigenvector of the matrix

$$\mathbf{B} = \mathbf{A} - 2\mathbf{I}$$

If so, what is the eigenvalue corresponding to $\mathbf{v}$?

Answer

Yes, $\mathbf{v}$ is also an eigenvector of $\mathbf{B}$ and the corresponding eigenvalue is $\lambda = 2$.

Problem 9.2

Without using MATLAB , find the eigenvalues and eigenvectors of the following matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Answer

$$\lambda_1 = 0 \quad ; \quad \lambda_2 = 1 \quad ; \quad \lambda_3 = 3$$
$$\mathbf{v}_1 = [1,\ 1,\ 1]^T \quad ; \quad \mathbf{v}_2 = [1,\ 0, -1]^T \quad ; \quad \mathbf{v}_3 = [1, -2, 1]^T$$

Problem 9.3

(a) Prove that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues. *Hint:* Look at the characteristic polynomials of the two matrices.

(b) Are the eigenvectors for $\mathbf{A}$ and $\mathbf{A}^T$ the same? Explain or show why or why not.

Answer

(a) To prove that $\mathbf{A}$ and $\mathbf{A}^T$ have the same eigenvalues, show that the characteristic polynomials of $\mathbf{A}$ and $\mathbf{A}^T$ are the same, which follows from the properties of determinants.

(b) Although the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^T$ are the same, the eigenvectors may not be the same. You may prove this by finding a simple example.

Problem 9.4

(a) **True** or **False**: If λ is an eigenvalue of $\mathbf{A}$ and if μ is an eigenvalue of $\mathbf{B}$, then $\lambda\mu$ is an eigenvalue of $\mathbf{AB}$. Justify your answer.

(b) **True** or **False**: If λ is an eigenvalue of $\mathbf{A}$ and if μ is an eigenvalue of $\mathbf{B}$, then $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$. Justify your answer.

Answer

(a) False

(b) False

Problem 9.5

If λ_i are the eigenvalues of $\mathbf{A}$ and μ_i are the eigenvalues of the matrix $\mathbf{B}$, then what is the product of the eigenvalues of the matrix $\mathbf{AB}$?

Answer

If α_i are the eigenvalues of $\mathbf{AB}$ then

$$\prod_i \alpha_i = \prod_i \lambda_i \prod_i \mu_i$$

Problem 9.6

Find the eigenvalues and a basis for the eigenspaces of $\mathbf{A}^{25}$ where

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Answer

The eigenvalues of $\mathbf{A}^{25}$ are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = -1$ and the eigenvectors are

$$\mathbf{v}_1 = [1,\ -1,\ 0] \quad ; \quad \mathbf{v}_2 = [1,\ 0,\ -1] \quad ; \quad \mathbf{v}_3 = [2, -1,\ 1]$$

Therefore, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are a basis for the eigenspace associated with $\lambda = 1$, and the vector $\mathbf{v}_3$ is a basis for the eigenspace of the eigenvalue $\lambda = -1$.

Problem 9.7

Suppose that matrices $\mathbf{A}$ and $\mathbf{B}$ are similar,

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$$

Determine whether or not each of the following statements are true or false, and give a proof or justification for your answer.

(a) Similar matrices $\mathbf{A}$ and $\mathbf{B}$ have the same rank.

(b) Similar matrices $\mathbf{A}$ and $\mathbf{B}$ have the same trace.

(c) Similar matrices $\mathbf{A}$ and $\mathbf{B}$ have the same nullity.

Answer

(a) True
(b) True
(c) True

Problem 9.8
The characteristic polynomial of a matrix $\mathbf{A}$ is

$$p(\lambda) = \lambda^4 - 3\lambda^3 + \lambda^2 + 6\lambda + 12$$

What is the determinant of $\mathbf{A}$?

Answer

The determinant is equal to 12.

Solutions to Regular Problems

Problem 9.9*
If $\mathbf{A}$ is an invertible matrix and $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, is $\mathbf{v}$ an eigenvector of $\mathbf{A}^{-1}$? If so, show why it is an eigenvector. If not, explain why it is not an eigenvector.

Solution

If $\mathbf{A}$ is an invertible matrix and $\mathbf{v}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{v}$ is also an eigenvector of $\mathbf{A}^{-1}$. This may be seen by noting that if

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

then multiplying on the left by $\mathbf{A}^{-1}$ we have

$$\mathbf{v} = \lambda\mathbf{A}^{-1}\mathbf{v}$$

Therefore,

$$\mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$$

so $\mathbf{v}$ is an eigenvector of $\mathbf{A}^{-1}$ and the corresponding eigenvalue is λ^{-1}.

Problem 9.10*
Let $\mathbf{A}$ be a matrix of ones and $\mathbf{B}$ a checkerboard matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Without using MATLAB

(a) Find the rank of each matrix.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{B}$.

Solution

(a) The rank of both of these matrices is easy to find. For $\mathbf{A}$, note that all four rows are the same. Therefore, there is only one independent row, and the rank of $\mathbf{A}$ is one. For $\mathbf{B}$, note that the first and third columns and the second and fourth columns are the same. Thus, the maximum that the rank of this matrix can be is two. Since the first and second columns are not multiples of each other, then they are linearly independent. Therefore, the rank of $\mathbf{B}$ is equal to two.

(b) To find the eigenvalues and eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, we can do a lot by inspection without any calculations. First consider the matrix $\mathbf{A}$. Since the rank is equal to one, there is only one non-zero eigenvalue. It is clear that $\mathbf{v} = [1,\ 1,\ 1,\ 1]^T$ is an eigenvector of $\mathbf{A}$ since $\mathbf{Av} = 4\mathbf{v}$, and the eigenvalue is four. In addition to $\lambda_1 = 4$, $\mathbf{A}$ has three other eigenvalues, all of which are equal to zero, $\lambda_2 = \lambda_3 = \lambda_4 = 0$. Since the columns of $\mathbf{A}$ are identical, it is not hard to find three linearly independent eigenvectors corresponding to this eigenvalue. These eigenvectors are solutions to the equation $\mathbf{Av} = 0$, and the following three vectors are one possible set of independent vectors (note that any linear combination of these eigenvectors will also be an eigenvector),

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Now let's consider the matrix $\mathbf{B}$. Since the rank is equal to two, $\mathbf{A}$ will have two non-zero eigenvalues, and two that are equal to zero. Again, if we are clever, we can find the eigenvalues and eigenvectors without doing any complicated math. It is not too hard to look at the form of the matrix and see that $\mathbf{v}_1 = [1,\ 1,\ 1,\ 1]^T$ will be an eigenvector, and the eigenvalue is $\lambda_1 = 2$. Similarly, we can observe that $\mathbf{v}_2 = [1,\ -1,\ 1,\ -1]^T$ will also be an eigenvector, and the eigenvalue in this case is $\lambda_2 = -2$. To find two linearly independent eigenvectors corresponding to $\lambda_3 = \lambda_4 = 0$ is again straightforward. We want to find a linear combination of the columns of $\mathbf{B}$ that will sum to zero. Clearly, the following two is one possible linearly independent set:

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Problem 9.11*

Find a matrix $\mathbf{A}$ whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$, with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solution

The key to this problem is to recall that a matrix with unique eigenvalues may be diagonalized as follows

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda}$$

where $\mathbf{P}$ is a matrix whose columns are the eigenvectors of $\mathbf{A}$ and $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues. Therefore, we may write

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1}$$

With

$$\mathbf{P} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

we have

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

Problem 9.12★

Find a basis for the eigenspaces of the following matrices

(a) $\mathbf{A}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Solution

(a) Since $\mathbf{A}_1$ is a diagonal matrix, the eigenvalues are equal to the values along the diagonal, so $\lambda_1 = 2$ and $\lambda_2 = 1$. Also, for a diagonal matrix, the eigenvectors are equal to the unit vectors $\mathbf{e}_i$, so

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

and it follows that $\mathbf{v}_1$ is a basis for the (one-dimensional) eigenspace corresponding to the eigenvalue $\lambda_1 = 2$ and $\mathbf{v}_2$ is a basis for the (one-dimensional) eigenspace corresponding to the eigenvalue $\lambda_2 = 1$.

(b) Matrix $\mathbf{A}_2$ is a lower triangular matrix so, just as with a diagonal matrix, the eigenvalues are equal to the values along the diagonal, so $\lambda_1 = 3$ and $\lambda_2 = -1$. To find the eigenvector corresponding to the first eigenvalue, we must find the solution to the homogeneous equations

$$(\mathbf{A} - 3\mathbf{I}) = \mathbf{0}$$

or

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

With x_2 a free variable, we see from the second equation that

$$8x_1 - 4x_2 = 0 \quad \Longrightarrow \quad x_1 = 0.5x_2$$

Therefore

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T$$

is an eigenvector corresponding to the eigenvalue $\lambda_1 = 3$ and is a basis for this eigenspace.

To find the eigenvector corresponding to the second eigenvalue, we must find the solution to the homogeneous equations

$$(\mathbf{A} + \mathbf{I}) = \mathbf{0}$$

or

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and we clearly see that

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T$$

is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$ and is a basis for this eigenspace.

(c) For $\mathbf{A}_3$, we need to find the eigenvalues from the characteristic polynomial

$$p(\lambda) = \det(\mathbf{A}_3 - \lambda\mathbf{I}) = 0$$

so, we need to evaluate

$$\det(\mathbf{A}_3 - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

Performing a cofactor expansion along the first row we have

$$-\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda) = (2-\lambda)[2-\lambda(3-\lambda)] = -(\lambda-2)^2(\lambda-1) = 0$$

and we see that $\mathbf{A}_3$ has an eigenvalue $\lambda = 2$ of multiplicity two and an eigenvalue $\lambda = 1$ of multiplicity one.

To find the eigenvectors associated with $\lambda = 2$ we need to find the set of solutions to the homogeneous equations

$$(\mathbf{A}_3 - 2\mathbf{I})\mathbf{v} = \mathbf{0}$$

or,

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we see that there are two linearly independent solutions to these equations which are given by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and these form a basis for the eigenspace associated with the eigenvalue $\lambda = 2$.

To find the eigenvector associated with $\lambda = 1$ we need to find the set of solutions to the homogeneous equations

$$(\mathbf{A}_3 - \mathbf{I})\mathbf{v} = \mathbf{0}$$

or,

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, we see by inspection that the eigenvector is

$$\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

and this vector forms a basis for the eigenspace associated with the eigenvalue $\lambda = 1$.

Problem 9.13★

Find the matrix $\mathbf{P}$ that diagonalizes each of the following matrices,

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Solution

(a) The eigenvalues of $\mathbf{A}_1$ are found to be $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$, and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad ; \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, a diagonalizing matrix is

$$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(b) The eigenvalues of $\mathbf{A}_1$ are found to be $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 1$, and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, a diagonalizing matrix is

$$\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$