CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #8

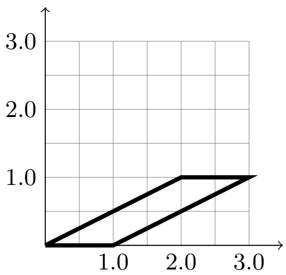
Problem 8.1

For each of the following transformations from R^2 to R^2 defined by the given matrix $\mathbf{A}$, determine what the effect of the transformation is on the unit square.

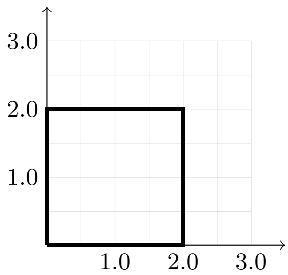
(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Answer

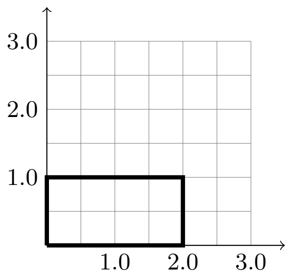
(a)



(b)



(c)



Problem 8.2

Find the matrix that performs a rotation of 45 degrees about the vector $\mathbf{v} = [1,\ 1,\ 1]^T$.

Answer

Normalizing the vector $\mathbf{v}$ so it is of unit length,

$$\mathbf{v} = [a,\ b,\ c]^T = \frac{1}{\sqrt{3}}[1,\ 1,\ 1]^T$$

and $\theta = \pi/4$, we have

$$\mathbf{A} = \begin{bmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta \\ ab(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta \\ ac(1-\cos\theta)-b\sin\theta & bc(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0.80474 & -0.31062 & 0.50588 \\ 0.50588 & 0.80474 & -0.31062 \\ -0.31062 & -0.31062 & 0.80474 \end{bmatrix}$$

Problem 8.3

Find a single matrix that performs the following operations in R^2.

(a) Reflection about the y-axis, then expansion by a factor of five in the x-direction, and then reflection about the line $x = y$.

(b) Rotation through an angle of thirty degrees about the origin, then shearing by a factor of -2 in the y-direction, and then expansion by a factor of three in the y-direction.

Answer

(a) $\mathbf{A}_1 = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}$ (b) $\mathbf{A}_1 = \begin{bmatrix} 0 & -1 \\ 5 & 0 \end{bmatrix}$ (c) $\mathbf{A}_1 = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 3-6\sqrt{3} & 6+3\sqrt{3} \end{bmatrix}$

Problem 8.4

Find the equation of the line in R^2 that is formed when the line

$$y = -4x + 3$$

is transformed by the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Answer

Find how two points on the line are mapped by the transformation, and then use the two-point form for an equation for a line,

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The result is

$$y = \tfrac{11}{16}x + \tfrac{3}{16}$$

Problem 8.5

Find the matrix for a shear in the x-direction that transforms a triangle with vertices $(0,0)$, $(2,1)$, and $(3,0)$ into a right triangle with right angle at the origin.

Answer

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Problem 8.6

Find the reflection of the point $(2,3)$ about the line

$$2x + 3y = 0$$

Answer

With $\cos\theta = 3/\sqrt{13}$, $\sin\theta = 2/\sqrt{13}$, and

$$\mathbf{P}_\theta = \begin{bmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \sin^2\theta \end{bmatrix}$$

the desired matrix transformation is

$$\mathbf{H}_\theta = (2\mathbf{P}_\theta - \mathbf{I}) = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

and the desired point is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{H}_\theta \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cos 2\theta + 3\sin 2\theta \\ 2\sin 2\theta - 3\cos 2\theta \end{bmatrix}$$

Problem 8.7

Find the equation of the line in R^2 that is produced when the line $y = 2x$ is transformed by each of the following transformations.

(a) Shear by a factor of three in the x-direction.

(b) Compression by a factor of $1/2$ in the y-direction.

(c) Reflection about the y-axis.

(d) Rotation of sixty degrees about the origin.

Answer

(a) $y = (2/7)x$

(b) $y = x$

(c) $y = -2x$

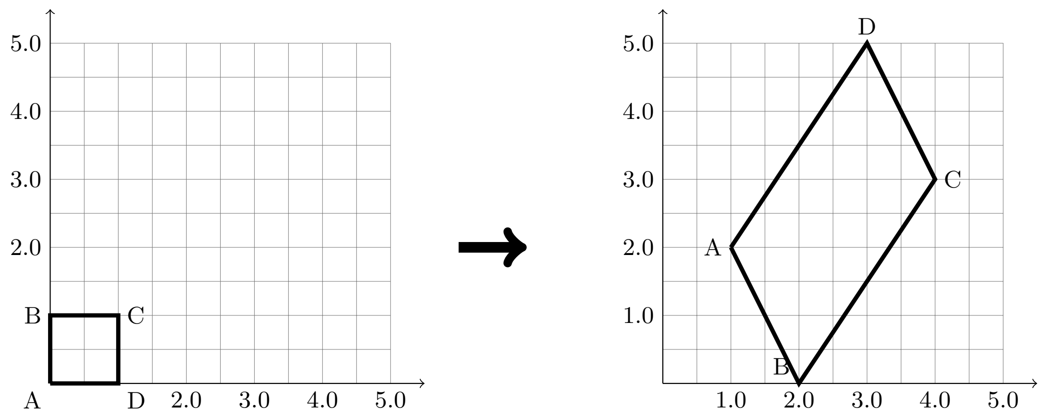
(d) $y = -\dfrac{8 + 5\sqrt{3}}{11}x$

Problem 8.8

Find the image of the unit square in R^2 under the affine transformation defined by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Answer



Solutions to Regular Problems

Problem 8.1⋆

In R^3, a shear in the $x-y$ direction with factor k is the matrix transformation that moves each point (x, y, z) parallel to the $x-y$ plane to a new position $(x+kz, y+kz, z)$. Find the matrix that performs a shear in the $x-y$ direction by a factor of k.

Solution

It is clear from the way that the shear is defined that the desired transformation is given by the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 8.2⋆

Determine whether or not the linear transformations defined by the matrices below are one-to-one.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$ (c) $\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Solution

(a) The two columns of $\mathbf{A}_1$ are linearly independent, so the dimension of the row space, which is the same as the dimension of the column space, is equal to two. Therefore, the dimension of the null space is zero, which means that the transformation is one-to-one.

(b) Since the dimension of the row space for $\mathbf{A}_2$ is equal to two, and $\mathbf{A}_2$ maps R^3 to R^2, then the dimension of the nullspace is equal to one. Therefore, this mapping is not one-to-one.

(c) As we have in part (a), the dimension of the row space is equal to three, and since $\mathbf{A}_3$ is a mapping from R^3 to R^5, then the dimension of the nullspace is zero, and this mapping is one-to-one.

Problem 8.3⋆

Find the matrix that performs a projection of a point in R^3 onto each of the following surfaces:

(a) The $y-z$ plane.

(b) The line through the origin defined by the equation

$$\mathbf{x} = k \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Solution

(a) $\mathbf{P}_{yz} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\mathbf{P}_x = \dfrac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ -1 & -3 & -1 \end{bmatrix}$

Problem 8.4*

The equation of a line in R^n that passes through the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$ is given by the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

where t is an arbitrary real number. If $\mathbf{x}_0 = \mathbf{0}$ then the line passes through the origin.

(a) Prove the following proposition:

> **Proposition**: Suppose that T is a linear transformation from R^n to R^n, and that
>
> $$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$
>
> is a line in R^n. Then the image of this line under the transformation T is either a line or a point. There are no other possibilities.

(b) Under what general set of conditions will the transformation T map a line to a point? Give an example of a non-zero transformation that maps a line to a point for $n = 3$.

(c) Prove that for any linear transformation, a triangle is mapped to another triangle.

Solution

(a) Each point in the line

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

is mapped by a linear transformation T that is represented by a matrix $\mathbf{A}$ as follows,

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}_0 + t\mathbf{A}\mathbf{v}$$

Let

$$\mathbf{x}_0' = \mathbf{A}\mathbf{x}_0 \quad \text{and} \quad \mathbf{v}' = \mathbf{A}\mathbf{v}$$

Then the line is mapped to a set of points defined by

$$\mathbf{y} = \mathbf{x}_0' + t\mathbf{v}'$$

which is an equation for a line, provided $\mathbf{v}' \neq \mathbf{0}$. If $\mathbf{v}' = \mathbf{0}$, then the line is mapped to a single point $\mathbf{x}_0'$.

(b) The line will map to a point if and only if $\mathbf{A}\mathbf{v} = \mathbf{0}$, which means that the vector $\mathbf{v}$ is in the null space of the transformation $\mathbf{A}$.

(c) Suppose that we have a triangle with vertices at P_1, P_2, and P_3 and suppose that these points are mapped to Q_1, Q_2, and Q_3 by a linear transformation $\mathbf{A}$. The line segment from P_1 to P_2 is mapped to a line segment from Q_1 to Q_2. Similarly, the line segment from P_2 to P_3 is mapped to a line segment from Q_2 to Q_3 and the line segment from P_3 back to P_1 is mapped to a line segment from Q_3 to Q_1. Therefore, it follows that the triangle with vertices P_1, P_2, and P_3 is mapped to a triangle with vertices Q_1, Q_2, and Q_3 by any linear transformation $\mathbf{A}$.

Problem 8.5⋆

Find the affine transformation that maps the triangle with vertices

$$\mathbf{p}_0 = (0,\ 1), \quad \mathbf{p}_1 = (1,\ 2), \quad \mathbf{p}_2 = (2,\ 0)$$

to a triangle with vertices

$$\mathbf{q}_0 = (1,\ 2), \quad \mathbf{q}_1 = (0,\ -1), \quad \mathbf{q}_2 = (3,\ 3)$$

Solution

We know that an affine transformation in R^3,

$$\begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

is uniquely defined by the mapping of three points. More specifically, the mapping from $\mathbf{p}_0$ to $\mathbf{q}_0$ is defined by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

or

$$a_2 + a_3 = 1$$
$$b_2 + b_3 = 2$$

Similarly for the mapping from $\mathbf{p}_1$ to $\mathbf{q}_1$ is defined by

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which we may write as

$$a_1 + 2a_2 + a_3 = 0$$
$$b_1 + 2b_2 + b_3 = -1$$

and finally, for the mapping from $\mathbf{p}_2$ to $\mathbf{q}_2$ is defined by

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

which gives us the final two equations,

$$2a_1 + a_3 = 3$$
$$2b_1 + b_3 = 3$$

Now, if we write Eqs. (1), (3), and (5) in matrix form we may solve for a_1, a_2, and a_3,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -4/3 \\ 7/3 \end{bmatrix}$$

Similarly, if we write Eqs. (2), (4), and (6) in matrix form we may solve for b_1, b_2, and b_3,

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -7/3 \\ 13/3 \end{bmatrix}$$

So, for the matrix that performs the desired affine transformation, we have

$$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & -4 & 7 \\ -2 & -7 & 13 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: It is always good to check your answer to make sure that $\mathbf{p}_0$ is mapped to $\mathbf{q}_0$, $\mathbf{p}_1$ is mapped to $\mathbf{q}_1$, and $\mathbf{p}_2$ is mapped to $\mathbf{q}_2$.