Chung-Ang University

Linear Algebra   Spring 2014

Problem Set #8

Assigned: May 15, 2014
Due Date: May 22, 2014

Reading: Anton, Sections 4.9 - 4.11 (pp. 247-282).

Problems: Given below are two sets of problems. The *Practice Problems* are for extra practice and you **do not** need to turn these in for grading. The starred problem, are the *Regular Problems* that are to be turned in for grading.

Practice Problems (Not to be turned in)

Problem 8.1

For each of the following transformations from R^2 to R^2 defined by the given matrix $\mathbf{A}$, determine what the effect of the transformation is on the unit square.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\mathbf{A}_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

Problem 8.2

Find the matrix that performs a rotation of 45 degrees about the vector $\mathbf{v} = [1,\ 1,\ 1]^T$.

Problem 8.3

Find a single matrix that performs the following operations in R^2.

(a) Reflection about the y-axis, then expansion by a factor of five in the x-direction, and then reflection about the line $x = y$.

(b) Rotation through an angle of thirty degrees about the origin, then shearing by a factor of -2 in the y-direction, and then expansion by a factor of three in the y-direction.

Problem 8.4

Find the equation of the line in R^2 that is formed when the line

$$y = -4x + 3$$

is transformed by the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$

Problem 8.5

Find the matrix for a shear in the x-direction that transforms a triangle with vertices $(0, 0)$, $(2, 1)$, and $(3, 0)$ into a right triangle with right angle at the origin.

Problem 8.6

Find the reflection of the point $(2, 3)$ about the line

$$2x + 3y = 0$$

Problem 8.7

Find the equation of the line in R^2 that is produced when the line $y = 2x$ is transformed by each of the following transformations.

(a) Shear by a factor of three in the x-direction.

(b) Compression by a factor of $1/2$ in the y-direction.

(c) Reflection about the y-axis.

(d) Rotation of sixty degrees about the origin.

Problem 8.8

Find the image of the unit square in R^2 under the affine transformation defined by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Regular Problems (To be turned in for grading)

Problem 8.1★

In R^3, a shear in the $x - y$ direction with factor k is the matrix transformation that moves each point (x, y, z) parallel to the $x - y$ plane to a new position $(x + kz, y + kz, z)$. Find the matrix that performs a shear in the $x - y$ direction by a factor of k.

Problem 8.2★

Determine whether or not the linear transformations defined by the matrices below are one-to-one.

(a) $\mathbf{A}_1 = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$ (b) $\mathbf{A}_2 = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{bmatrix}$ (c) $\mathbf{A}_3 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Problem 8.3★

Find the matrix that performs a projection of a point in R^3 onto each of the following surfaces:

(a) The $y - z$ plane.

(b) The line through the origin defined by the equation

$$\mathbf{x} = k \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Problem 8.4★

The equation of a line in R^n that passes through the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$ is given by the equation

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

where t is an arbitrary real number. If $\mathbf{x}_0 = \mathbf{0}$ then the line passes through the origin.

(a) Prove the following proposition:

Proposition: Suppose that T is a linear transformation from R^n to R^n, and that

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

is a line in R^n. Then the image of this line under the transformation T is either a line or a point. There are no other possibilities.

(b) Under what general set of conditions will the transformation T map a line to a point? Give an example of a non-zero transformation that maps a line to a point for $n = 3$.

(c) Prove that for any linear transformation, a triangle is mapped to another triangle.

Problem 8.5*

Find the affine transformation that maps the triangle with vertices

$$\mathbf{p}_0 = (0,\ 1), \quad \mathbf{p}_1 = (1,\ 2), \quad \mathbf{p}_2 = (2,\ 0)$$

to a triangle with vertices

$$\mathbf{q}_0 = (1,\ 2), \quad \mathbf{q}_1 = (0,\ -1), \quad \mathbf{q}_2 = (3,\ 3)$$

Problem 8.6*

In the classic game of *Lights Out!* you are given board with a 5×5 array of lights that are in some initial state, on or off, and the problem is to find a sequence of lights to push so that eventually all lights are turned off. We saw that there will be a solution if and only if the initial state of the lights is *orthogonal* to the two vectors that span the two-dimensional nullspace of a 25×25 matrix **A**. (Note: In this problem, you will need to use MATLAB .)

(a) Suppose that we design a the *Lights Out!* game and use a board that has an 8×8 array of lights. When is the game solvable? In other words, what are the conditions on the initial state of lights, if necessary, that must be imposed in order for there to be a solution?

(b) If the only light that is initially on is the one in the upper right-hand corner, find a sequence of moves that will turn all of the lights off.

(c) Repeat part (a) for a board that has an 9×9 array of lights.