CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #7

Practice Problems

Problem 7.1

Let $\mathbf{A}$ be an $m \times n$ matrix. What is the smallest possible value for its nullity?

Answer

Nullity$(A) \geq n - \min(m, n)$

Problem 7.2

(a) Give an example of a 3×3 matrix whose column space is a plane through the origin in R^3.

(b) What type of geometric object is the null space of your matrix in part (a)?

(c) What type of geometric object is the row space of your matrix in part (a)?

Answer

(a) Any 3×3 matrix that has two linearly independent columns will have a column space that is a plane passing through the origin.

(b) If a 3×3 matrix has a column space that is a plane passing through the origin, then the dimension of the null space will be equal to one. Therefore, the nullspace will be a line passing through the origin.

(c) The row space will be one-dimensional and, therefore, will be a line passing through the origin.

Problem 7.3

True or **False**: If $\mathbf{A}$ is an $n \times n$ matrix, then the nullspace is equal to the left nullspace.

Answer

The answer to this problem is *False.* In this problem it is important to recognize that even though two vector spaces have the same dimension, they may not be the same vector space.

Problem 7.4

Find the dimension and construct a basis for the four subspaces associated with the following matrix,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer

1. The dimension of the column space of $\mathbf{A}$ is equal to three,

$$\dim(\mathcal{R}(\mathbf{A})) = 3$$

and a basis for this subspace is the last three columns of $\mathbf{A}$.

2. The dimension of the row space is equal to the dimension of the column space, so

$$\dim(\mathcal{R}(\mathbf{A}^T)) = 3$$

The first three rows of $\mathbf{A}$ form a basis for the row space.

3. The dimension of the null space is equal to one, and a basis for this space is the vector $\mathbf{v} = [1,\ 0,\ 0,\ 0]^T$.
4. The dimension of the left null space is equal to one and a basis for the null space is the vector $\mathbf{v} = [0,\ 0,\ 0,\ 1]^T$.

Problem 7.5

What is the dimension of the nullspace of the 4×4 identity matrix?

Answer

The dimension is zero.

Regular Problems

Problem 7.1*

Show that if the product of two matrices is the zero matrix, $\mathbf{AB} = \mathbf{0}$, then the column space of $\mathbf{B}$ is *contained* in the nullspace of $\mathbf{A}$. What about the row space of $\mathbf{A}$ and the left nullspace of $\mathbf{B}$? Can you make a similar statement?

Solution

If the product of two matrices is the zero matrix,

$$\mathbf{AB} = \mathbf{0}$$

then for each column vector $\mathbf{b}_i$ of $\mathbf{B}$,

$$\mathbf{A}\mathbf{b}_i = \mathbf{0}$$

Since any vector $\mathbf{v}$ in the column space of $\mathbf{b}$ is a linear combination of the columns of $\mathbf{B}$

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_m\mathbf{b}_m$$

then

$$\mathbf{Av} = a_1\mathbf{A}\mathbf{b}_1 + a_2\mathbf{A}\mathbf{b}_2 + \cdots + a_m\mathbf{A}\mathbf{b}_m = \mathbf{0}$$

Therefore, every vector in the column space of $\mathbf{B}$ is in the null space of $\mathbf{A}$, which means that *the column space of* $\mathbf{A}$ *is contained in the null space of* $\mathbf{B}$.

If $\mathbf{AB} = \mathbf{0}$, taking the transpose of both sides of the equation we have

$$\mathbf{B}^T\mathbf{A}^T = \mathbf{0}$$

Therefore, similar to what we found above, it follows that the column space of $\mathbf{A}^T$ (the row space of $\mathbf{A}$) is contained in the null space of $\mathbf{A}^T$ (the left null space of $\mathbf{A}$). Therefore, if $\mathbf{AB} = \mathbf{0}$ then *the row space of* $\mathbf{A}$ *is contained in the left null space of* $\mathbf{B}$.

Problem 7.2⋆

Construct a matrix whose nullspace is spanned by $[1,\ 0,\ 1]^T$.

Solution

The null space is the orthogonal complement to the row space, and since $n = 3$ (the number of columns of $\mathbf{A}$), then the dimension of the row space is equal to two. Therefore, since the row space is the orthogonal complement to the row space, we need to find two linearly independent vectors that are orthogonal to the vector $\mathbf{v} = [1,\ 0,\ 1]^T$. Two such vectors are

$$\mathbf{r}_1 = [0,\ 1,\ 0]^T \quad ; \quad \mathbf{r}_2 = [1,\ 0,\ -1]^T$$

Therefore, a matrix whose nullspace is spanned by the given vector $\mathbf{v}$ is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Problem 7.3⋆

Listed below is the output of MATLAB showing the row echelon form $\mathbf{R}$ of a matrix $\mathbf{A}$:

```
A =
     2     1     5     8     4     2     2
     1     0     2     4     1     1     1
    -1    -3    -5    -4    -7     1     1
     1     0     2     5     3     1     0
    -1    -2    -4    -1     1     3     0

R =
     1     0     2     0    -7     0     4
     0     1     1     0     2     0     0
     0     0     0     1     2     0    -1
     0     0     0     0     0     1     1
     0     0     0     0     0     0     0
```

Find the dimension and construct a basis for each of the four subspaces that are associated with the matrix $\mathbf{A}$.

Solution

We want to find the dimension of each of the four subspaces of $\mathbf{A}$ and find bases for these subspaces. First, we note that there are four linearly independent rows in the reduced row echelon form of $\mathbf{A}$, so the rank of the matrix is equal to four. Therefore, the dimension of the column space and the row space is

$$\dim(\mathcal{R}(\mathbf{A})) = \dim(\mathcal{R}(\mathbf{A}^T)) = 4$$

Next, since $\mathbf{A}$ is a 5×7 matrix, then

$$\dim(\mathcal{N}(\mathbf{A})) = 7 - \mathrm{rank}(\mathbf{A}) = 3$$

$$\dim(\mathcal{N}(\mathbf{A}^T)) = 5 - \mathrm{rank}(\mathbf{A}) = 1$$

Since columns 1, 2, 4, and 6 in the reduced echelon form of $\mathbf{A}$ are linearly independent, then so are the same columns in $\mathbf{A}$. Therefore, a basis for the column space is

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} ; \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ 2 \end{bmatrix} ; \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ 4 \\ -4 \\ 5 \\ -1 \end{bmatrix} ; \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Since the row space is not changed by elementary row operations, and since the first three rows of $\mathbf{R}$ are linearly independent, then they form a basis for $\mathcal{R}(\mathbf{A}^T)$,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -7 \\ 0 \\ 4 \end{bmatrix}; \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

To find the null space, want to find the solutions to the homogeneous linear equations,

$$\mathbf{A}\mathbf{x} = 0$$

We see from the row echelon form of $\mathbf{R}$ that x_3, x_5 and x_7 are free variables. Therefore, let

$$x_3 = s; \quad x_5 = t; \quad x_7 = u$$

Finding the solution to the homogeneous equations with $s = 1$ and $t = u = 0$ we find the first basis vector. Then, with $t = 1$ and $s = u = 0$ we find the second basis vector. Finally, with $u = 1$ and $s = t = 0$ we have the third basis vector. These vectors are

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ -2 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Finally, to find a basis for the left null space, we need to find a basis for the null space of $\mathbf{A}^T$. For the reduced row echelon form of $\mathbf{A}$ we find

```
R =

     1     0     0     0     4
     0     1     0     0   -10
     0     0     1     0     2
     0     0     0     1     3
     0     0     0     0     0
     0     0     0     0     0
     0     0     0     0     0
```

To solve the equation $\mathbf{A}^T\mathbf{x} = \mathbf{0}$ we see that the solution is

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 10 \\ -2 \\ -3 \\ 1 \end{bmatrix}$$

which is a basis for the left null space (recall that the dimension is equal to one).

Problem 7.4⋆

For each of the following problems, determine whether or not the linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

is consistent and, if it is, find the dimension of the solution set, which is equivalent to the number of *free parameters*.

(a) $\mathbf{A}$ is 4×4 and Rank$(\mathbf{A}) = 4$, Rank$([\mathbf{A}; \mathbf{b}]) = 4$.

(b) $\mathbf{A}$ is 5×5 and Rank$(\mathbf{A}) = 3$, Rank$([\mathbf{A}; \mathbf{b}]) = 4$.

(c) $\mathbf{A}$ is 6×6 and Rank$(\mathbf{A}) = 3$, Rank$([\mathbf{A}; \mathbf{b}]) = 3$.

(d) $\mathbf{A}$ is 5×7 and Rank$(\mathbf{A}) = 3$, Rank$([\mathbf{A}; \mathbf{b}]) = 4$.

(e) $\mathbf{A}$ is 7×4 and Rank$(\mathbf{A}) = 3$, Rank$([\mathbf{A}; \mathbf{b}]) = 3$.

Solution

We have a linear system $\mathbf{Ax} = \mathbf{b}$.

(a) If $\mathbf{A}$ is 4×4 and Rank$(\mathbf{A}) = 4$, then $\mathbf{A}$ has full rank and is invertible. Therefore, the linear system $\mathbf{Ax} = \mathbf{b}$ has a unique answer.

(b) If Rank$(\mathbf{A}) = 3$ and Rank$([\mathbf{A} : \mathbf{b}]) = 4$, then $\mathbf{b}$ is linearly independent of the columns of $\mathbf{A}$. This means that there is *no solution* to the linear system, and the equations are inconsistent.

(c) If $\mathbf{A}$ is 6×6 and Rank$(\mathbf{A}) = 3$, then $\mathbf{A}$ is not invertible. However, since Rank$([\mathbf{A} : \mathbf{b}])$ is also equal to three, then $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{A}$. This means that $\mathbf{b}$ is in the column space of $\mathbf{A}$ and the equations are consistent. Since the dimension of the null space of $\mathbf{A}$ is equal to three, then there are three parameters. In other words, there is a three-dimensional solution space.

(d) If Rank$(\mathbf{A}) = 3$ and Rank$([\mathbf{A}|\mathbf{b}]) = 4$, then $\mathbf{b}$ is linearly independent of the columns of $\mathbf{A}$. This means that there is *no solution* to the linear system, and the equations are inconsistent.

(e) If $\mathbf{A}$ is 7×4 and Rank$(\mathbf{A}) = 3$ then the dimension of the row space is equal to three and the dimension of the null space is equal to one. Since Rank$([\mathbf{A}|\mathbf{b}]) = 3$, which is the same as the rank of $\mathbf{A}$, then $\mathbf{b}$ is in the column space of $\mathbf{A}$. Therefore, the linear system has a solution, and the dimension of the solution space is one (the dimension of the null space).

Problem 7.5⋆

Given two matrices $\mathbf{A}$ and $\mathbf{B}$, what is the relationship between the subspaces given below? In other words, are they equal, is one subspace contained within the other, or is there no relationship at all?

(a) $\mathcal{N}(\mathbf{AB})$ and $\mathcal{N}(\mathbf{B})$.

(b) $\mathcal{R}(\mathbf{AB})$ and $\mathcal{R}(\mathbf{A})$.

(c) $\mathcal{N}((\mathbf{AB})^T)$ and $\mathcal{N}(\mathbf{A}^T)$.

(d) $\mathcal{R}((\mathbf{AB})^T)$ and $\mathcal{R}(\mathbf{B}^T)$.

Solution

(a) It should be clear that the nullspace of $\mathbf{B}$ is contained in the nullspace of $\mathbf{AB}$,

$$\mathcal{N}(\mathbf{AB}) \supseteq \mathcal{N}(\mathbf{B})$$

To see this, note that if a vector $\mathbf{v}$ is in the nullspace of $\mathbf{B}$, then $\mathbf{Bv} = \mathbf{0}$ and $\mathbf{A}(\mathbf{Bv} = \mathbf{0}$. So any vector in the nullspace of $\mathbf{B}$ will be in the nullspace of $\mathbf{AB}$. However, it is possible that the nullspace of $\mathbf{AB}$ is larger than the nullspace of $\mathbf{B}$, since there may be vectors $\mathbf{v}$ that are not in the nullspace of $\mathbf{B}$ and $\mathbf{A}(\mathbf{Bv}) = \mathbf{0}$. For example, note that if

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then the nullspace of $\mathbf{B}$ is the set of all vectors

$$\mathbf{v} = k[1,\ 0]^T$$

and since $\mathbf{AB} = \mathbf{0}$ then the null space of $\mathbf{AB}$ is equal to all of R^2.

(b) Suppose that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times k$ matrix. Then,

$$T_B : R^k \longrightarrow R^n$$
$$T_A : R^n \longrightarrow R^m$$
$$T_{AB} : R^k \longrightarrow R^n \longrightarrow R^m$$

If the dimension of the left nullspace of $\mathbf{B}$ is zero, then the dimension of the column space is equal to n, and the column space is equal to R^n. In this case, the column space of $\mathbf{AB}$ is the same as the column space of $\mathbf{A}$. If, however, the dimension of the left nullspace of $\mathbf{B}$ is not equal to zero, then the column space of $\mathbf{B}$ will be a subspace of R^n. In this case, the column space of $\mathbf{AB}$ may be smaller than the column space of $\mathbf{A}$. So, we have

$$\mathcal{R}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{A})$$

As an example, let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Note that

$$\mathcal{R}(\mathbf{A}) = R^2$$

and

$$\mathcal{R}(\mathbf{AB}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

which is a subspace of the column space of $\mathbf{A}$.

(c) Since $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ then

$$\mathcal{N}((\mathbf{AB})^T) = \mathcal{N}(\mathbf{B}^T\mathbf{A}^T)$$

and we know from part (a) that $\mathcal{N}(\mathbf{AB}) \supseteq \mathcal{N}(\mathbf{B})$. Therefore,

$$\mathcal{N}(\mathbf{B}^T\mathbf{A}^T) \supseteq \mathcal{N}(\mathbf{A}^T)$$

and it follows that

$$\mathcal{N}((\mathbf{AB})^T) \supseteq \mathcal{N}(\mathbf{A}^T)$$

(d) Again, since $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ then

$$\mathcal{R}((\mathbf{AB})^T) = \mathcal{R}(\mathbf{B}^T\mathbf{A}^T)$$

and we know from part (b) that

$$\mathcal{R}(\mathbf{B}^T\mathbf{A}^T) \subseteq \mathcal{R}(\mathbf{B}^T)$$

Therefore,

$$\mathcal{R}((\mathbf{AB})^T) \subseteq \mathcal{R}(\mathbf{B}^T)$$

Problem 7.6*

Given two matrices $\mathbf{A}$ and $\mathbf{B}$, what is the relationship between

(a) The rank of $\mathbf{AB}$ and the rank of $\mathbf{A}$.

(b) The rank of $\mathbf{AB}$ and the rank of $\mathbf{B}$.

(c) The dimension of the nullspace of $\mathbf{AB}$ and the dimension of the nullspace of $\mathbf{B}$

Solution

(a) As we saw in the previous problem, $\mathcal{R}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{A})$. Therefore, for the rank of $\mathbf{AB}$ we have

$$r(\mathbf{AB}) = \dim(\mathcal{R}(\mathbf{AB})) \leq \dim(\mathcal{R}(\mathbf{A})) = r(\mathbf{A})$$

Thus, the rank of $\mathbf{AB}$ will be less than or equal to the rank of $\mathbf{A}$.

(b) Using the property that the rank of a matrix and its transpose are the same, $r(\mathbf{A}) = r(\mathbf{A}^T)$ we have

$$r(\mathbf{AB}) = r((\mathbf{AB}^T)) = r(\mathbf{B}^T\mathbf{A}^T) \leq r(\mathbf{B}^T) = r(\mathbf{B})$$

where the inequality follows from the result of part (a). Therefore, the rank of $\mathbf{AB}$ will be less than or equal to the rank of $\mathbf{B}$. Note that what we see from parts (a) and (b) is that the rank of a matrix can never be increased by multiplying it by another matrix, either on the left or on the right.

(c) From part (a) of the previous problem, we saw that

$$\mathcal{N}(\mathbf{AB}) \supseteq \mathcal{N}(\mathbf{B})$$

Therefore, the dimension of the nullspace of $\mathbf{AB}$ must be greater than or equal to the dimension of the nullspace of $\mathbf{B}$