CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #5

Problem 5.1

Which of the following are *subspaces* of R^4 (give a reason for your answer).

(a) The set of all vectors of the form $[a,\ 0,\ a,\ 0]^T$.

(b) The set of all vectors of the form $[a,\ b,\ a+b,\ a-b]^T$.

(c) The set of all vectors of the form $[a,\ 0,\ b,\ 0]^T$.

(d) The set of all vectors of the form $[1,\ 0,\ a,\ 0]^T$.

Solution

(a) The set of all vectors of the form $[a,\ 0,\ a,\ 0]$ is a vector space, which may be established as follows. Let $\mathbf{u} = [a,\ 0,\ a,\ 0]$ and $\mathbf{v} = [b,\ 0,\ b,\ 0]$ then

$$\mathbf{u} + \mathbf{v} = [a+b,\ 0,\ a+b,\ 0]$$

is in the set so it is closed under addition. Also, for any scalar c

$$c\mathbf{u} = [ca,\ 0,\ ca,\ 0]$$

is also in the set, so it is closed under scalar multiplication. Therefore, the set of all vectors of the form $[a,\ 0,\ a,\ 0]$ is a vector space.

(b) The set of all vectors of the form $[a,\ b,\ a+b,\ a-b]$ is a vector space. Let $\mathbf{u} = [a,\ b,\ a+b,\ a-b]$ and $\mathbf{v} = [c,\ d,\ c+d,\ c-d]$ then

$$\mathbf{u} + \mathbf{v} = [a+c,\ b+d,\ a+b+c+d,\ a+c-b-d] = [a+c,\ b+d,\ (a+c)+(b+d),\ (a+c)-(b+d)]$$

is in the set, so it is closed under addition. Also, for any scalar c

$$c\mathbf{u} = [ca,\ cb,\ c(a+b),\ c(a-b)]$$

is also in the set, so it is closed under scalar multiplication.

(c) The set of all vectors of the form $[a,\ 0,\ b,\ 0]^T$ is a subspace of R^4 because it is closed under addition and scalar multiplication as can be shown in a manner to what was done in part (a).

(d) The set of all vectors of the form $[1,\ 0,\ a,\ 0]^T$ is **not** a subspace of R^4 because the zero vector is not in the set.

Problem 5.2

Which of the following are *subspaces* of R^∞ (give a reason for your answer).

(a) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 0,\ a,\ 0,\ a,\ 0\cdots]^T$$

(b) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 1,\ a,\ 1,\ a,\ 1\cdots]^T$$

(c) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 2a,\ 4a,\ 16a, \cdots,\ 2^n a, \cdot]^T$$

(d) The set of all sequences in R^∞ whose components are equal to zero from some point on.

Solution

(a) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 0,\ a,\ 0,\ a,\ 0 \cdots]^T$$

is a subspace of R^∞. This follows from the fact that if

$$\mathbf{v}_1 = [a,\ 0,\ a,\ 0,\ a,\ 0 \cdots]^T$$
$$\mathbf{v}_2 = [b,\ 0,\ b,\ 0,\ b,\ 0 \cdots]^T$$
$$\mathbf{v}_1 + \mathbf{v}_2 = [a+b,\ 0,\ a+b,\ 0,\ a+b,\ 0 \cdots]^T$$

and if c is a scalar, then

$$c\mathbf{v} = [ca,\ 0,\ ca,\ 0,\ ca,\ 0 \cdots]^T$$

is in the set. Since the set is closed under addition and scalar multiplication, it is a subspace of R^∞.

(b) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 1,\ a,\ 1,\ a,\ 1 \cdots]^T$$

is not a subspace because the zero vector is not in the set.

(c) The set of all sequences $\mathbf{v}$ in R^∞ that are of the form

$$\mathbf{v} = [a,\ 2a,\ 4a,\ 16a, \cdots,\ 2^n a, \cdot]^T$$

is a subspace for the same reasons given in part (a)).

(d) The set of all sequences in R^∞ whose components are equal to zero from some point on is a subspace, again for the same reasons given in part (a).

Problem 5.3

Which of the following are *subspaces* of $M_{n\times n}$ (give a reason for your answer).

(a) The set of all invertible matrices.

(b) The set of all diagonal matrices.

Solution

To determine whether or not the given sets are subspaces of $M_{n\times n}$, it is necessary to determine whether or not the sets are closed under addition and scalar multiplication.

(a) The set of all invertible matrices is **not** a subspace of $M_{n\times n}$ because the sum of two matrices is not necessarily invertible. For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are both invertible, but the sum is not.

(b) The set of all diagonal matrices is a subspace of $M_{n\times n}$. Clearly, the addition of two diagonal matrices is a diagonal matrix, and when a diagonal matrix is multiplied by a constant, it remains a diagonal matrix. Therefore, diagonal matrices are closed under addition and scalar multiplication and are therefore a subspace of $M_{n\times n}$.

Problem 5.4

Which of the following are *subspaces* of P_4, the vector space of all polynomials of degree 4 or less.

(a) All polynomials of the form $p(x) = ax$.

(b) All polynomials that have real roots.

Solution

To show that a subset of P_4 is a subspace, it is necessary to show that it is **closed** under addition and scalar multiplication.

(a) All polynomials of the form $\mathbf{p} = ax$ is a subspace of P_4 since adding two polynomials of the form $\mathbf{p} = ax$ clearly results in a polynomial of the same form, and the same is true when multiplying a polynomial $\mathbf{p} = ax$ by a constant. So this space is closed under addition and scalar multiplication and, therefore, is a subspace of P_4.

(b) All polynomials that have real roots is **not** a subspace of P_4. For example, let $\mathbf{p}_1 = 1+x$ and $\mathbf{p}_2 = x^2$ be polynomials in P_4. Although both $\mathbf{p}_1$ and $\mathbf{p}_2$ have real roots, $\mathbf{p}_1 + \mathbf{p}_2$ does not.

Problem 5.5

What is the dimension of the following vector spaces?

(a) The vector space of all $n \times n$ upper triangular matrices

(b) All vectors of the form $[a, b, 0, 0]^T$

(c) All polynomials of the form $p(x) = ax^2 + bx^4$.

(d) The vector space of all 2×2 matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a + b + c = 0$.

Solution

(a) The dimension of the vector space of all $n \times n$ upper triangular matrices is equal to the number of elements on or above the diagonal of the matrix, which is equal to

$$\sum_{k=1}^{n} k = \tfrac{1}{2}(n)(n+1)$$

(b) The dimension of the vector space of all vectors of the form $[a, b, 0, 0]^T$ is two. One possible basis is $\mathbf{v}_1 = [1,\ 0,\ 0,\ 0]^T$, $\mathbf{v}_1 = [0,\ 1,\ 0,\ 0]^T$.

(c) The dimension of the vector space of all polynomials of the form $p(x) = ax^2 + bx^4$ is equal to two. A basis is $\mathbf{p}_1 = x^2$ and $\mathbf{p}_2 = x^4$. .

(d) The dimension of the vector space of all 2×2 matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a + b + c = 0$ is equal to three.

Problem 5.6

Which of the following sets of vectors form a basis for R^3?

(a) $\mathbf{v}_1 = [1,\ 0,\ 0]^T$, $\mathbf{v}_2 = [1,\ 1,\ 0]^T$, $\mathbf{v}_3 = [1,\ 1,\ 1]^T$.

(b) $\mathbf{v}_1 = [1,\ 0,\ 0]^T$, $\mathbf{v}_2 = [0,\ 1,\ 0]^T$, $\mathbf{v}_3 = [0,\ 0,\ 1]^T$, $\mathbf{v}_4 = [1,\ 0,\ 1]^T$.

(c) $\mathbf{v}_1 = [1,\ 2,\ 3]^T$, $\mathbf{v}_2 = [3,\ 2,\ 1]^T$, $\mathbf{v}_3 = [1,\ 2,\ 4]^T$

Solution

(a) The vectors $\mathbf{v}_1 = [1,\ 0,\ 0]^T$, $\mathbf{v}_2 = [1,\ 1,\ 0]^T$, $\mathbf{v}_3 = [1,\ 1,\ 1]^T$ are a basis for R^3. One way to see this is by noting that $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$ so $\mathbf{v}_1$ is linearly independent of $\mathbf{v}_1$. Similarly, since the third component of $\mathbf{v}_3$ is non-zero and the third component of $\mathbf{v}_1$ and $\mathbf{v}_2$ is equal to zero, then $\mathbf{v}_3$ is linearly independent of $\mathbf{v}_1$ and $\mathbf{v}_2$. Since we have three linearly independent vectors in $\mathbf{R}^3$ then they must form a basis for $\mathbf{R}^3$.

(b) The set of vectors $\mathbf{v}_1 = [1,\ 0,\ 0]^T$, $\mathbf{v}_2 = [0,\ 1,\ 0]^T$, $\mathbf{v}_3 = [0,\ 0,\ 1]^T$, $\mathbf{v}_4 = [1,\ 0,\ 1]^T$ is **not** a basis for R^3 because the number of vectors in a basis for R^3 must be equal to three.

(c) The vectors $\mathbf{v}_1 = [1,\ 2,\ 3]^T$, $\mathbf{v}_2 = [3,\ 2,\ 1]^T$, $\mathbf{v}_3 = [1,\ 2,\ 4]^T$ form a basis for R^3 since $\mathbf{A} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is row-equivalent to $\mathbf{I}_3$. Therefore, these vectors are linearly independent, and any set of three linearly independent vectors in R^3 will form a basis for R^3.

Problem 5.7

Which of the following vectors in P_3 are linearly independent? For those sets that are linearly dependent, which vectors in the set are dependent upon the others?

(a) $\mathbf{p}_1(x) = 1,\ \ \mathbf{p}_2(x) = 1 + x,\ \ \mathbf{p}_3(x) = 1 + x + x^2,\ \ \mathbf{p}_4(x) = 1 + x + x^2 + x^3.$

(b) $\mathbf{p}_1(x) = 1 + x,\ \ \mathbf{p}_2(x) = 1 - x,\ \ \mathbf{p}_3(x) = 1 + x^2 + x^3.$

(c) $\mathbf{p}_1(x) = 1,\ \ \mathbf{p}_2(x) = 1 + x,\ \ \mathbf{p}_3(x) = 1 + x^2,\ \ \mathbf{p}_4(x) = x^3.$

Solution

We are given a set of vectors in P_3 and we want to determine whether or not they are linear independent. If they are dependent, we want to express the vector as a linear combination of the given vectors.

(a) The vectors $\mathbf{p}_1 = 1,\ \ \mathbf{p}_2 = 1 + x,\ \ \mathbf{p}_3 = 1 + x + x^2,\ \ \mathbf{p}_4 = 1 + x + x^2 + x^3$ are linearly independent since, as we progress from $\mathbf{p}_1$ to $\mathbf{p}_4$ progressively higher order terms are added.

(b) The vectors $\mathbf{p}_1 = 1 + x,\ \ \mathbf{p}_2 = 1 - x,\ \ \mathbf{p}_3 = 1 + x^2 + x^3$ are linearly independent, which may be shown as follows. Clearly, $\mathbf{p}_1$ is linearly independent of $\mathbf{p}_2$ since one is not a constant times the other. Next, since $\mathbf{p}_3$ contains the terms x^2 and x^3, which are not in either $\mathbf{p}_1$ or $\mathbf{p}_2$, then $\mathbf{p}_3$ is linearly independent of $\mathbf{p}_1$ and $\mathbf{p}_2$. It then follows that the three vectors are linearly independent.

(c) The vectors $\mathbf{p}_1 = 1,\ \ \mathbf{p}_2 = 1 + x,\ \ \mathbf{p}_3 = 1 + x^2,\ \ \mathbf{p}_4 = x^3$ are linearly independent since, as we progress from $\mathbf{p}_1$ to $\mathbf{p}_4$ progressively higher order terms are added.

Problem 5.8

You are given a set of ten vectors in R^{20}. How would you use MATLAB to determine whether or not these vectors are linearly independent?

Solution

If we are given a set of ten vectors in R^{20}, we may use MATLAB to test for linear independence of these vectors as follows. First, form the matrix $\mathbf{A} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{20}]$. Then, find the reduced echelon form of $\mathbf{A}$ using the `rref` command in MATLAB . If the reduced echelon form has no rows that are equal to zero, then the vectors are linearly independent.