CHUNG-ANG UNIVERSITY

Linear Algebra   Spring 2014

Solutions to Problem Set #4

Practice Problems

Problem 4.1

For what values of α, if any, is the vector $\mathbf{v}^T = [8\alpha,\ -2]$ parallel to the vector $\mathbf{u}^T = [4,\ -1]$?

Answer

$\alpha = 1$

Problem 4.2

Let $\mathbf{u}$ and $\mathbf{v}$ be the following vectors:

$$\mathbf{u} = [\ 2,\ 1,\ 0,\ 1,\ -1]^T; \quad \mathbf{v} = [-2,\ 3,\ 1,\ 0,\ \ 2]^T$$

and let

$$\mathbf{c} = [-8,\ 8,\ 3,\ -1,\ 7]^T$$

Find scalars a and b such that

$$a\mathbf{u} + b\mathbf{v} = \mathbf{c}$$

Answer

$b = 3; \quad a = -4 + b = -1$

Problem 4.3

Let P be the point $(2,\ 3,\ -2)$ and Q the point $(7,\ -4,\ 1)$.

(a) Find the midpoint of the line segment connecting P and Q.

(b) Find the point on the line segment connecting P and Q that is 3/4 of the way from P to Q.

Answer

(a) The midpoint between P and Q is

$$R = \tfrac{1}{2}(P + Q) = \tfrac{1}{2}(9, -1, -1)$$

(b) The point that lies 3/4 of the way from P to Q is

$$R = \tfrac{1}{4}(P + 3Q) = \tfrac{1}{4}(23, -9, 1)$$

Problem 4.4

Let $\mathbf{u}$ and $\mathbf{v}$ be the following vectors,

$$\mathbf{u} = [\ 2,\ -1,\ -4,\ 1,\ 0,\ 6,\ -3,\ 1]^T$$
$$\mathbf{v} = [-2,\ -1,\ 2,\ 3,\ 7,\ 2,\ -5,\ 1]^T$$

(a) Find the (Euclidean) distance between $\mathbf{u}$ and $\mathbf{v}$.

(b) Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

(c) Find a vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Answer

(a) The distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u},\mathbf{v}) = \sqrt{(4)^2+(0)^2+(-6)^2+(-2)^2+(-7)^2+(4)^2+(2)^2+(0)^2} = \sqrt{125}$$

(b) The cosine of the angle is

$$\cos\theta = \frac{20}{\sqrt{68}\sqrt{97}}$$

(c) The following vector is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{w} = [0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1]^T$$

Regular Problems (To be turned in for grading)

Problem 4.1⋆

Find a unit vector that is orthogonal to the vectors

$$\mathbf{u} = [\ 1,\ 2,\ -1,\ 3]^T; \quad \mathbf{v} = [-1,\ 0,\ 4,\ -1]^T$$

Solution

We want to find a vector $\mathbf{w}$ that is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, or

$$\mathbf{w}\cdot\mathbf{u} = 0; \qquad \mathbf{w}\cdot\mathbf{v} = 0$$

If we let $\mathbf{w} = [w_1,\ w_2,\ w_3,\ w_4]^T$, then we may write these two equations in matrix form as follows,

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ -1 & 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To find a solution to these two equations, we form the augmented matrix and then reduce it to row echelon form,

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ -1 & 0 & 4 & -1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{bmatrix}$$

So, w_3 and w_4 are free variables. To find a vector that is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, let $w_3 = 0$ and $w_4 = 1$ (these can be any values as long as both are not equal to zero). Then, from the second equation that corresponds to the second row in the augmented matrix, we have

$$2w_2 + 3w_3 + 2w_4 = 0 \Longrightarrow w_2 = -1$$

and the equation corresponding to the first row of the augmented matrix we have

$$w_1 + 2w_2 - w_3 + 3w_4 = 0 \Longrightarrow w_1 = -1$$

Therefore,

$$\mathbf{w} = [-1,\ -1,\ 0,\ 1]^T$$

Since we want a unit vector that is orthogonal to **w** and **v**, then we need to divide by the norm of this vector,

$$\|\mathbf{w}\| = \sqrt{3}$$

which gives

$$\mathbf{w} = \frac{1}{\sqrt{3}}[-1,\ -1,\ 0,\ 1]^T$$

Problem 4.2★

Find a point-normal form equation for a plane that passes through the point

$$P = (1,\ 1,\ 4)$$

and is normal to the vector

$$\mathbf{n} = [\ 1,\ 9,\ 8]^T$$

Solution

The point-normal form equation for a plane that passes through a point $P = (x_1, x_2, \ldots, x_p)$ that is normal to the vector $\mathbf{n} = [n_1, n_2, \ldots, n_p]^T$ is

$$n_1(x - x_1) + n_2(x - x_2) + \cdots + n_p(x - x_p) = 0$$

Therefore, the point-normal form equation for a plane that passes through the point

$$P = (1,\ 1,\ 4)$$

that is normal to the vector

$$\mathbf{n} = [\ 1,\ 9,\ 8]^T$$

is

$$(x_1 - 1) + 9(x_2 - 1) + 8(x_3 - 4) = 0$$

or

$$x_1 + 9x_2 + 8x_3 = 42$$

Problem 4.3

In each part below you are given a pair of linear equations that represent planes in R^3. For each, determine whether or not the planes are parallel, orthogonal, or neither.

(a)

$$\begin{aligned} x - \ \ 4y - 3z &= 2 \\ 3x - 12y - 9z &= 7 \end{aligned}$$

(b)

$$\begin{aligned} 2x + 3y - \ \ z &= 6 \\ x - 4y + 4z &= 1 \end{aligned}$$

(c)

$$\begin{aligned} 3x + \ \ y - 2z &= 6 \\ 2x - 4y + \ \ z &= 0 \end{aligned}$$

Solution

(a) Note that the first plane is orthogonal to the vector

$$\mathbf{n}_1 = [1,\ -4,\ -3]^T$$

while the second plane is orthogonal to

$$\mathbf{n}_2 = [3,\ -12,\ -9]^T$$

Since

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 + 48 + 27 = 78$$

the planes are not orthogonal. However, since

$$\|\mathbf{n}_1\|\|\mathbf{n}_2\| = \sqrt{26}\sqrt{234} = 78$$

then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \|\mathbf{n}_1\|\|\mathbf{n}_2\|$$

Therefore, $\mathbf{n}_1$ and $\mathbf{n}_2$ point in the same direction, so the planes are parallel.

(b) The first plane is orthogonal to the vector

$$\mathbf{n}_1 = [2,\ 3,\ -1]^T$$

while the second plane is orthogonal to

$$\mathbf{n}_2 = [1,\ -4,\ 4]^T$$

Since

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 12 - 4 = -14$$

then the planes are not orthogonal, and since

$$\|\mathbf{n}_1\|\|\mathbf{n}_2\| = \sqrt{14}\sqrt{33} \neq \mathbf{n}_1 \cdot \mathbf{n}_2$$

then the planes are not parallel.

(c) The first plane is orthogonal to the vector

$$\mathbf{n}_1 = [3,\ 1,\ -2]^T$$

while the second plane is orthogonal to

$$\mathbf{n}_2 = [2,\ -4,\ 1]^T$$

Since

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 6 - 4 - 2 = 0$$

then the two planes are orthogonal.

Problem 4.4*

Find $\|\mathrm{proj}_{\mathbf{a}}\mathbf{u}\|$, the projection of $\mathbf{u}$ onto the vector $\mathbf{a}$, when

$$\mathbf{u} = [1,\ -1,\ 0]^T; \quad \mathbf{a} = [2,\ 0,\ 1]^T$$

Solution

The projection of $\mathbf{u}$ onto $\mathbf{a}$ is a vector in the direction of $\mathbf{a}$ with a length that makes the vector $\mathbf{u} - \mathrm{proj}_{\mathbf{a}}(\mathbf{u})$ orthogonal to $\mathbf{a}$. The projection is given by

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{\langle \mathbf{u}, \mathbf{a} \rangle}{\|\mathbf{a}\|^{\mathbf{2}}}\mathbf{a}$$

With $\langle \mathbf{u}, \mathbf{a} \rangle = 2$ and $\|\mathbf{a}\|^2 = 5$, then

$$\mathrm{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{2}{5}\mathbf{a} = \frac{2}{5}[2,\ 0,\ 1]^T$$

and

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\|^2 = \frac{4}{25}\|[2,\ 0,\ 1]\|^2 = \frac{4}{25}5$$

so

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{2}{\sqrt{5}}$$

Problem 4.5★

Find the component of the vector $\mathbf{u}$ along the vector $\mathbf{a}$ and the vector component of $\mathbf{u}$ that is orthogonal to $\mathbf{a}$ when

(a) $\mathbf{u} = [3,\ -1,\ 2]^T$ and $\mathbf{a} = [1,\ 3,\ 0]^T$.

(b) $\mathbf{u} = [5,\ 0,\ -3,\ 7]^T$ and $\mathbf{a} = [2,\ 1,\ -1,\ -1]^T$.

Solution

(a) The component of $\mathbf{u} = [3,\ -1,\ 2]^T$ along the vector $\mathbf{a} = [1,\ 3,\ 0]^T$ is the projection of $\mathbf{u}$ onto $\mathbf{a}$, which is

$$\text{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{\langle \mathbf{u}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}\mathbf{a}$$

However, note that

$$\langle \mathbf{u}, \mathbf{a} \rangle = 0$$

which says that $\mathbf{u}$ and $\mathbf{a}$ are orthogonal. Therefore, there is no component of $\mathbf{u}$ along $\mathbf{a}$

(b) With

$$\langle \mathbf{u}, \mathbf{a} \rangle = 6$$

and

$$\|\mathbf{a}\|^2 = 7$$

it follows that the projection of $\mathbf{u}$ onto $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}}(\mathbf{u}) = \frac{\langle \mathbf{u}, \mathbf{a} \rangle}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{6}{7}[2,\ 1,\ -1,\ -1]^T$$

The component of $\mathbf{u}$ that is orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}(\mathbf{u}) = [5,\ 0,\ -3,\ 7]^T - \frac{6}{7}[2,\ 1,\ -1,\ -1]^T = \frac{1}{7}[23,\ -6,\ -15,\ 55]^T$$

As a check, you can verify that these two vectors are orthogonal.

Problem 4.6★

Find the distance between the point $P = (2,\ 1,\ 3)$ and the plane

$$3x - y - 2z = 6$$

Solution

Given a hyperplane in R^3 defined by the following equation

$$\mathbf{n}^T\mathbf{x} + c = a_1x_1 + a_2x_2 + a_3x_3 + c = 0$$

and given any point $\mathbf{p} = (p_1, p_2, p_4)$ in R^n, the distance from $\mathbf{p}$ to the plane is

$$D = \frac{|a_1p_1 + a_2p_2 + a_3p_3 + c|}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

We are given a plane defined by the equation

$$3x - y - 2z = 6$$

Therefore, the distance of the point $P = (2,\ 1,\ 3)$ to the plane is

$$D = \frac{|(2)(3) + (1)(-1) + (3)(-2) - 6|}{\sqrt{3^2 + (-1)^2 + (-2)^2}} = \frac{7}{\sqrt{14}}$$

Problem 4.7*

Find the distance between the two parallel planes

$$\begin{aligned} 2x - y - z &= 5 \\ -4x + 2y + 2z &= 12 \end{aligned}$$

Solution

To find the distance between two parallel planes, all we need to do is select on point in one of the planes, and find the distance from that point to the other plane. So, we can clearly see that $P = (0,\ 0,\ 6)$ is a point in the second plane, and the distance from this point to the first plane is

$$D = \frac{|(2)(0) + (-1)(0) + (-1)(6) - 5|}{\sqrt{2^2 + (-1)^2 + (-1)^2}} = \frac{11}{\sqrt{6}}$$