CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #3

Problem 3.1

In this problem, you will look at the problem of polynomial interpolation using MATLAB or Octave.

(a) Find a tenth-order interpolating polynomial that passes through the points $(-5,-10)$, $(-4,-5)$, $(-3,2)$, $(-2,-5)$, $(-1,6)$, $(0,8)$, $(1,1)$, $(2,-9)$, $(3,1)$, $(4,2)$, and $(5,-1)$.

(b) Plot the data points and the interpolating polynomial to see if the polynomial passes through the given points.

(c) Find a second degree polynomial (if one exists) that passes through the points $(-2,-7.6)$, $(-1,-7.8)$, $(1,-1.0)$, $(2,6.0)$, and $(3,15.4)$. If the interpolating polynomial exists, make a plot that contains both the data points and the interpolating polynomial. If the interpolating polynomial does not exist, make a plot of the data points and explain why there can be no second degree polynomial that passes through the points.

(d) Find a second degree polynomial (if one exists) that passes through the points $(-3,1)$, $(-1,4)$, $(1,2)$, and $(2,6)$. If the interpolating polynomial exists, make a plot that contains both the data points and the interpolating polynomial. If the interpolating polynomial does not exist, make a plot of the data points and explain why there can be no second degree polynomial that passes through the points.

(e) Find two different interpolating polynomials of degree three that pass through the points $(-3,-24)$, $(-1,8)$, and $(2,-4)$. Make a plot that contains the data points and both of the interpolating polynomials.

(f) Now let's find the polynomial that produces the *best fit* to the function

$$y = 2^x$$

Choose five points between $x=-1$ and $x=3$ that lie on this curve, and find a fourth-order polynomial that passes through these points.

(g) Plot the interpolating polynomial along with the function $y=2^x$ and see how close the values of the polynomial are to the function between $x=-1$ and $x=3$. Describe what happens when the polynomial is compared to the function for values of x that are outside the interval $[-1,3]$. This problem illustrates the important issue of "goodness of fit." In fitting a polynomial through a set of points, the underlying assumption is that there is a polynomial relationship between the variables, or one that is approximately polynomial. If this is not the case, then the interpolating polynomials may not be a good solution for representing (modeling) the data.

Solution

(a) We want to find a tenth-order interpolating polynomial that passes through the points $(-5,-10)$, $(-4,-5)$, $(-3,2)$, $(-2,-5)$, $(-1,6)$, $(0,8)$, $(1,1)$, $(2,-9)$, $(3,1)$, $(4,2)$, and $(5,-1)$. The following sequence of MATLAB commands will solve this problem,

```
>> x = [-5 : 5]';
>> y = [ -10 -5 -2 -5 6 8 1 -9 1 2 -1]';
>> M=[x.^(10), x.^9, x.^8, x.^7, x.^6, x.^5, x.^4, x.^3, x.^2, x, ones(size(x)), y];
>> R=rref(M);  % Put the augmented matrix in reduced row echelon form.
>> p=R(:,12)   % Extract the last column of R to give the solution.
p =
    3.5494e-004
   -1.2676e-004
   -1.6873e-002
    6.4815e-003
    2.2850e-001
   -1.1377e-001
   -5.6831e-001
    9.4353e-001
   -4.1437e+000
   -3.3361e+000
    8.0000e+000
```

The coefficients of the tenth-order polynomial are contained in the vector p. As noted in the assignment, it is important to check the answer to make sure that the polynomial passes through the given points. The answer may be checked as follows,

```
>> yp=polyval(p,x)
yp =
    -10.00000
     -5.00000
      2.00000
     -5.00000
      6.00000
      8.00000
      1.00000
     -9.00000
      1.00000
      2.00000
     -1.00000
```

so, to within five decimal places, the polynomial passes through the given points.

(b) Next we plot the data points to see that the interpolating polynomial passes through the given points and to see what the interpolation looks like. The MATLAB code to plot the polynomial is

```
>>xp=linspace(-5,5);
>>yp=polyval(p,xp); % The vector p is the solution found in part (a).
>>plot(x,y,'ro',xp,yp,'-')
```

The plot is shown in Figure 1. From this plot, we clearly see the oscillations that occur between the points. Often, polynomial interpolation is done in order to extrapolate a given set of data points to values outside a given interval. Therefore, it is illustrative to see how this polynomial behaves outside the interval $[-5, 5]$. Shown in the Fig. 2 is a plot of the polynomial over the interval $[-6, 6]$ and we see that it increases very rapidly in both directions.

(c) Here we want to see if we can find a second degree polynomial that passes through the points $(-2, -7.6)$, $(-1, -7.8)$, $(1, -1.0)$, $(2, 6.0)$, and $(3, 15.4)$. Using MATLAB , we write

```
>> x = [-2 -1 1 2 3]';
>> y = [-7.6 -7.8 -1.0 6.0 15.4]';
>> M=[x.^2, x, ones(size(x)), y];
>> R=rref(M)
ans =
    1.00000  0.00000  0.00000  1.20000
    0.00000  1.00000  0.00000  3.40000
    0.00000  0.00000  1.00000 -5.60000
    0.00000  0.00000  0.00000  0.00000
    0.00000  0.00000  0.00000  0.00000
```

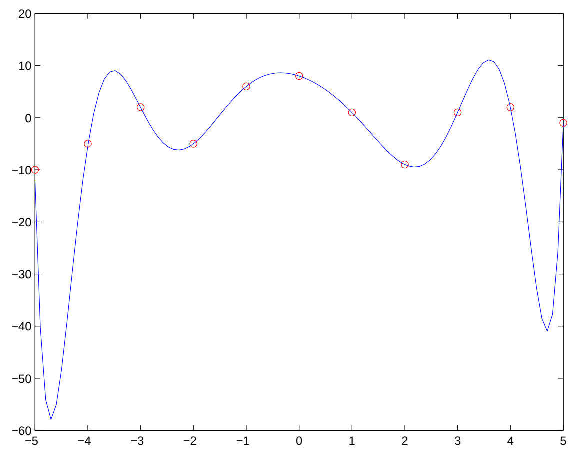

Figure 1: The tenth-order interpolating polynomial.

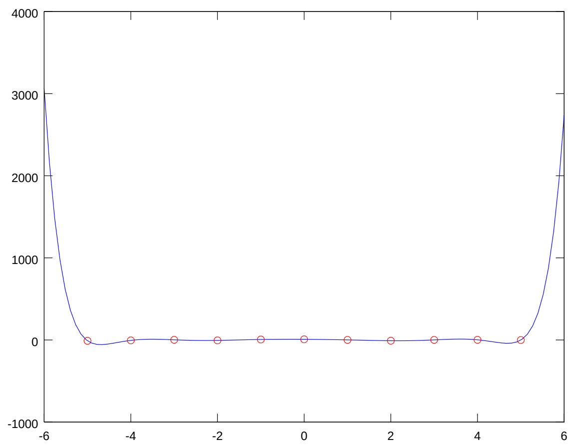

Figure 2: Extrapolation of the tenth-order interpolating polynomial.

So, although we have five equations in only three unknowns, the equations are consistent, and so there is a second-order polynomial that passes through the given points. The polynomial is

$$p(x) = 1.2x^2 + 3.4x - 5.6$$

A plot of this polynomial along with the given points can be created as follows

```
>>xp=linspace(-1,4);
>>yp=polyval(p(1:3),xp);
>>plot(x,y,'ro',xp,yp,'-')
```

The plot is shown in Figure 3 and, as we see, the polynomial passes through the given points.

(d) Repeating the previous part using the points $(-3, 1)$, $(-1, 4)$, $(1, 2)$, and $(2, 6)$ we find

```
>> x = [-3 -1 1 2]';
>> y = [ 1  4 2 6]';
>> M=[x.^2, x, ones(size(x)), y];
>> R=rref(M)
ans =
    1  0  0  0
    0  1  0  0
```

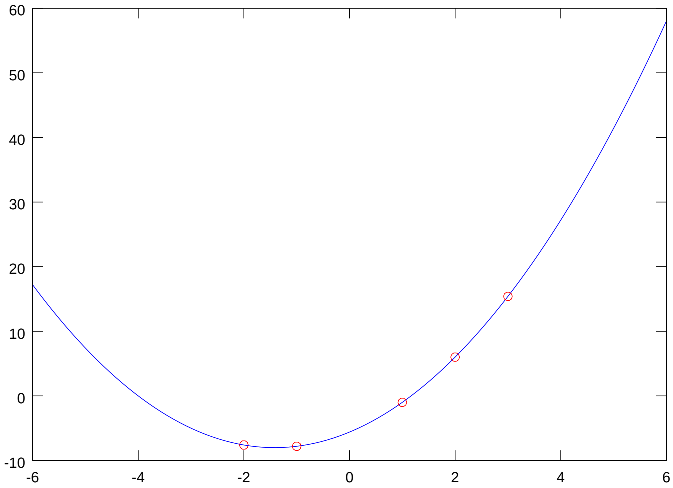

Figure 3: A second-order polynomial passing through the given points.

```
    0   0   1   0
    0   0   0   1
```

We see that after the sequence of elementary row reductions, the last equation becomes $0 = 1$, and so these equations are inconsistent and no solution exists, i.e., there is no second-order polynomial that passes through the given points.

(e) We are now given the following set of points, $(-3, -24)$, $(-1, 8)$, and $(2, -4)$. Again using MATLAB we can find a third-order polynomial that passes through these points. Note: a third-order polynomial has four coefficients and, since we are only given three points, we have fewer equations than unknowns so we should have an infinite number of solutions.

```
>> x = [-3  -1  2 ]';
>> y = [-24  8 -4 ]';
>> M=[x.^3, x.^2, x, ones(size(x)), y];
>> R=rref(M)
ans =
    1.00000  0.00000  0.00000  0.16667   2.00000
    0.00000  1.00000  0.00000  0.33333   0.00000
    0.00000  0.00000  1.00000 -0.83333 -10.00000
```

So, with our third-order polynomial,

$$p(x) = ax^3 + bx^2 + cx + d$$

the equations corresponding to the reduced echelon form matrix above are

$$\begin{aligned} a \qquad\quad + \ 0.16667d &= 2.00000 \\ b \quad\ + \ 0.33333d &= 0.00000 \\ c \ - \ 0.83333d &= -10.00000 \end{aligned}$$

where d is a free variable. Since we are free to set d equal to any real number, if we set $d = 0$ then

$$c = -10, \quad b = 0, \quad a = 2$$

and the polynomial is

$$p_1(x) = 2x^3 - 10x$$

If, on the other hand, we set $d = 10$ then

$$c = -5/3, \quad b = -10/3, \quad a = 1/3$$

and the polynomial is

$$p_2(x) = \tfrac{1}{3}x^3 - \tfrac{10}{3}x^2 - \tfrac{4}{3}x + 10$$

Plotting both of these polynomials along with the given points using the following sequence of MATLAB commands

```
>>xp=linspace(-1,4);
>>yp=polyval(p(1:3),xp);
>>plot(x,y,'ro',xp,yp,'-')
```

produces the plot shown in Figure 4. Clearly, we may generate an infinite number of different polynomials that pass through these points. Which one is best?

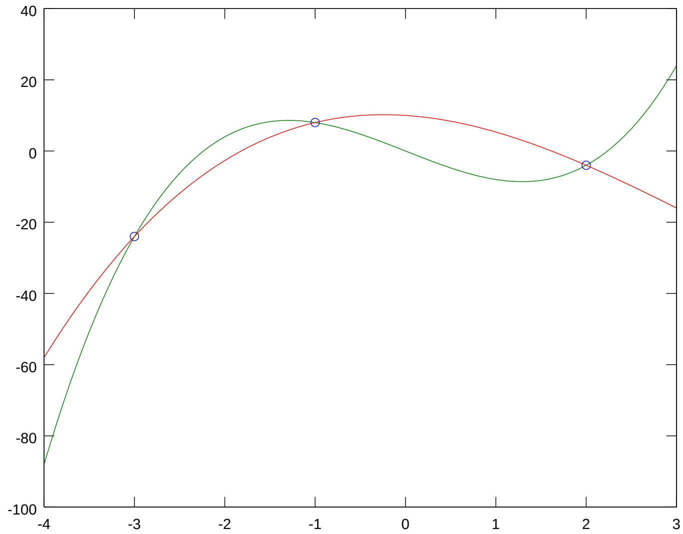

Figure 4: Two third-order polynomials passing through the given points, $p_1(x) = 2x^3 - 10x$ (green) and $p_2(x) = \frac{1}{3}x^3 - \frac{10}{3}x^2 - \frac{4}{3}x + 10$ (red).

(f) Now let's find the polynomial that produces the *best fit* to the function

$$y = 2^x$$

We will choose five equally-spaced points between $x = -1$ and $x = 3$ that line on this curve. They are

$$(-1, 1/2), \quad (0, 1), \quad (1, 2), \quad (2, 4), \quad (3, 8)$$

The fourth-order polynomial is found as follows,

```
>> x = [-1 0 1 2 3 ]';
>> y = [0.5 1 2 4 8 ]';
>> M=[x.^4, x.^3, x.^2, x, ones(size(x)), y];
>> R=rref(M);
>> p=R(:,6)
p =
    0.020833
    0.041667
    0.229167
    0.708333
    1.000000
```

so the polynomial is

$$p(x) = 0.020833x^4 + 0.041667x^3 + 0.229167x^2 + 0.708333x + 1$$

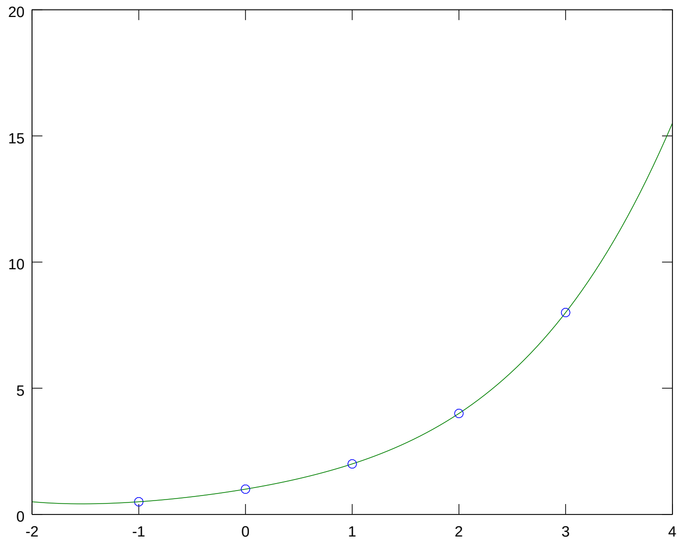

Figure 5: Plot of a fourth-order polynomial passing through five points of the function $f(x) = 2^x$.

(g) A plot of the fourth-order polynomial along with the five points is shown in Figure 5 and, as we see, the polynomial passes through the given points.

If we plot the fourth-order polynomial over the range $[-10, 10]$, which contains many points outside the range used to fit the function $f(x) = 2^x$ we see in Figure 6 that the polynomial and the function $f(x)$ are quite different. This problem illustrates the important issue of "goodness of fit." In fitting a polynomial through a set of points, the underlying assumption is that there is a polynomial relationship between the variables, or one that is approximately polynomial. If this is not the case, then the interpolating polynomials may not be a good solution for representing (modeling) the data.

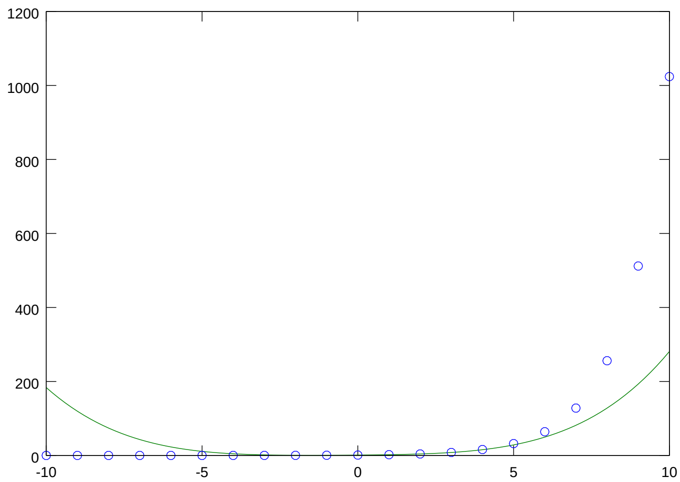

Figure 6: Plot showing the poor extrapolation that is performed by the fourth-order polynomial.