CHUNG-ANG UNIVERSITY

Linear Algebra Spring 2014

Solutions to Problem Set #2

Answers to Practice Problems

Problem 2.1

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be matrices of the following sizes;

$\mathbf{A}$	$\mathbf{B}$	$\mathbf{C}$	$\mathbf{D}$	$\mathbf{E}$
(3×1)	(3×6)	(6×2)	(2×6)	(1×3)

For each of the following, determine whether or not the given expression is defined. In other words, are the matrices of the correct size so that the given expression is a valid one. For those that are defined, determine the size of the resulting matrix.

(a) $\mathbf{B}^T(\mathbf{A} + \mathbf{E}^T)$

(b) $(\mathbf{C}^T + \mathbf{D})\mathbf{B}^T$

(c) $(\mathbf{B}\mathbf{D}^T)\mathbf{C}^T$

Answer

(a) Expression is defined, and is equal to a 6×1 vector.

(b) Expression is defined, and is a 2×3 matrix.

(c) Expression is defined, and the result is a 3×6 matrix.

Problem 2.2

Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}$$

and evaluate each of the following expressions.

(a) $\mathbf{A}(\mathbf{B}\mathbf{C})$

(b) $\mathrm{Tr}(4\mathbf{E}^T - \mathbf{D})$

(c) $\mathrm{Tr}(\mathbf{F}\mathbf{F}^T)$

Answer

(a) $\mathbf{A}(\mathbf{B}\mathbf{C}) = \begin{bmatrix} 58 & 22 \\ -50 & 226 \end{bmatrix}.$

(b) $\mathrm{Tr}(4\mathbf{E}^T - \mathbf{D}) = 4(3) - 1 = 11.$

(c) $\mathrm{Tr}(\mathbf{F}\mathbf{F}^T) = 3a^2 + 4b^2 + 2c^2$

Problem 2.3

Find all values of k, if any, that satisfy the following equation

$$[2 \;\; 2 \;\; k] \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$$

Answer

$k = -2, \quad -10$

Problem 2.4

A matrix is said to be an *orthogonal matrix* if its transpose is the same as its inverse. Thus, a matrix $\mathbf{A}$ is orthogonal if

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

or

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}$$

Show that

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix.

Answer

To show this, just form the transpose of $\mathbf{A}$, multiply it by $\mathbf{A}$ and show that the product is equal to the identity matrix.

Solutions to Regular Problems

Problem 2.1⋆

A matrix $\mathbf{B}$ is said to be a *Square Root* of a matrix $\mathbf{A}$ if

$$\mathbf{B}\mathbf{B} = \mathbf{A}$$

(a) Find *two* square roots of

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Hint: You can begin by noting that since $\mathbf{A}$ is symmetric, then the square root is probably also symmetric as well. So, let

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and then multiply out the equation $\mathbf{B}\mathbf{B} = \mathbf{A}$ and try to solve for a, b, and c by equating the coefficients on each side of the equation.

(b) How many different square roots can you find if

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reason.

Solution

(a) To find the square root(s) of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

we begin by noting that since $\mathbf{A}$ is symmetric, then the square root is probably also symmetric as well. So, let

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Forming the product $\mathbf{BB}$ and setting this equal to $\mathbf{A}$ we have

$$\mathbf{BB} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a^2+b^2 & b(a+c) \\ b(a+c) & b^2+c^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Now, we can solve for a, b, and c by setting equating the values of the coefficients in $\mathbf{A}$ and $\mathbf{BB}$. Since

$$\begin{aligned} a^2+b^2 &= 2 \\ c^2+b^2 &= 2 \end{aligned}$$

then

$$a^2 = c^2 \Longrightarrow a = \pm c$$

However, note that since

$$b(a+c) = 2$$

then $a = -c$ is not a solution since this would lead to

$$\mathbf{BB} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & b^2+c^2 \end{bmatrix}$$

So, $a = c$ and it follows that

$$b(a+c) = 2ab = 2 \Longrightarrow ab = 1$$

Finally, the solution to the pair of equations

$$\begin{aligned} a^2+b^2 &= 1 \\ ab &= 1 \end{aligned}$$

are

$$a = b = 1 \quad \text{and} \quad a = b = -1$$

so the square roots of $\mathbf{A}$ are

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$\mathbf{B}_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

(b) We can find four square roots of

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

They are

$$\mathbf{B}_1 = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} \sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$$

$$\mathbf{B}_3 = \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & 3 \end{bmatrix} \qquad \mathbf{B}_4 = \begin{bmatrix} -\sqrt{5} & 0 \\ 0 & -3 \end{bmatrix}$$

(c) The question as to whether or not a matrix will always have at least one square root is a difficult one. However, if we look at a few cases of some simple 2×2 matrices, it is not difficult to find one that does not have a square root. Consider the following matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

If $\mathbf{B}$ is the square root of $\mathbf{A}$ and

$$\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then we must have

$$\begin{aligned} a^2 + bc &= 0 \\ b(a+d) &= 1 \\ c(a+d) &= 0 \\ bc + d^2 &= 0 \end{aligned}$$

In order for the second equation to hold, $a + d \neq 0$.

Problem 2.2★

In real arithmetic, we know that

$$(a+b)^2 = a^2 + 2ab + b^2$$

Does this relationship also hold if a and b are matrices, i.e., is it always true that

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Explain why or why not.

Solution

We need to be careful when multiplying matrices because we must preserve the order in which they are multiplied. So let's expand $(\mathbf{A}+\mathbf{B})^2$ while being careful not to interchange the order of matrix products,

$$(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$$

So, we see that $(\mathbf{A}+\mathbf{B})^2$ will not be equal to $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ unless $\mathbf{BA} = \mathbf{AB}$, and this, in general, is not true.

Problem 2.3★

Determine whether or not the following matrices are invertible (do not use MATLAB). Note that here you are to *show* whether or not the matrix is invertible. You do not need to find the inverse (if it exists).

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Solution

(a) One way to check to see if $\mathbf{A}$ is invertible is to put it in reduced row echelon form. If the reduced row echelon form is equivalent to the identity matrix, then it is invertible. So, we have

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the matrix is invertible.

(b) As in part (a), we put $\mathbf{A}$ into reduced row echelon form,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

so $\mathbf{A}$ is invertible.

Problem 2.4★

A square matrix is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$.

(a) Show that if $\mathbf{A}$ is idempotent, then so is $\mathbf{I} - \mathbf{A}$.

(b) Show that if $\mathbf{A}$ is idempotent, then $2\mathbf{A} - \mathbf{I}$ is *invertible* and is its own inverse, i.e.,

$$(2\mathbf{A} - \mathbf{I})^{-1} = 2\mathbf{A} - \mathbf{I}$$

Solution

(a) If $\mathbf{A}$ is an idempotent matrix, then

$$\begin{aligned}(\mathbf{I} - \mathbf{A})^2 &= (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) \\ &= \mathbf{I} - \mathbf{A} - \mathbf{A} - \mathbf{A}^2 \\ &= \mathbf{I} - \mathbf{A}\end{aligned}$$

and so $\mathbf{I} - \mathbf{A}$ is also idempotent.

(b) If $\mathbf{A}$ is idempotent, then

$$\begin{aligned}(2\mathbf{A} - \mathbf{I})^2 &= (2\mathbf{A} - \mathbf{I})(2\mathbf{A} - \mathbf{I}) \\ &= 4\mathbf{A}^2 - 2\mathbf{A} - 2\mathbf{A} + \mathbf{I} \\ &= \mathbf{I}\end{aligned}$$

Therefore, $2\mathbf{A} - \mathbf{I}$ is equal to its own inverse.

Problem 2.5★

A square matrix $\mathbf{A}$ is said to be *nilpotent* if $\mathbf{A}^k = \mathbf{0}$ for some positive integer k, and the smallest number k for which this is true is said to be the *degree* of the matrix.

(a) Is the following matrix nilpotent? If so, what is its degree?

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 5 & 7 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Repeat part (a) for the following matrix,

$$\mathbf{B} = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

(c) Show that if $\mathbf{A}$ is nilpotent of degree n, then $\mathbf{I} - \mathbf{A}$ is invertible and

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$$

Hint: First show that for any integer n,

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}) = \mathbf{I} - \mathbf{A}^n$$

(Note that the scalar form of this formula is the same one that is used to derive the geometric series expansion).

Solution

(a) If you enter this matrix into MATLAB you will find that $\mathbf{A}^4 = \mathbf{0}$ but $\mathbf{A}^3 \neq \mathbf{0}$. Therefore, $\mathbf{A}$ is nilpotent of degree 4.

(b) Entering the matrix $\mathbf{B}$ into MATLAB you will again find that $\mathbf{B}^2 = \mathbf{0}$, so $\mathbf{B}$ is nilpotent of degree 2.

(c) Note that for any integer n, it is always true that

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}) = \mathbf{I} - \mathbf{A}^n$$

(this trick is how the formula for the geometric series is derived). So, if there is a value of n for which $\mathbf{A}^n = \mathbf{0}$, then

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}) = \mathbf{I}$$

which implies that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$$

The condition that $\mathbf{A}^n = \mathbf{0}$, of course, means that $\mathbf{A}$ is nilpotent of degree n.