Chung-Ang University

Linear Algebra Spring 2014

Problem Set #2

Assigned: March 11, 2014
Due Date: March 20, 2014

Reading: Anton, Sections 1.3 - 1.7 (pp. 25-73).

Problems: Given below are two sets of problems. The *Practice Problems* are for extra practice and you **do not** need to turn these in for grading. The starred problem, are the *Regular Problems* that are to be turned in for grading.

Practice Problems (Not to be turned in)

Problem 2.1

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be matrices of the following sizes;

$\mathbf{A}$	$\mathbf{B}$	$\mathbf{C}$	$\mathbf{D}$	$\mathbf{E}$
(3×1)	(3×6)	(6×2)	(2×6)	(1×3)

For each of the following, determine whether or not the given expression is defined. In other words, are the matrices of the correct size so that the given expression is a valid one. For those that are defined, determine the size of the resulting matrix.

(a) $\mathbf{B}^T(\mathbf{A}+\mathbf{E}^T)$

(b) $(\mathbf{C}^T+\mathbf{D})\mathbf{B}^T$

(c) $(\mathbf{B}\mathbf{D}^T)\mathbf{C}^T$

Problem 2.2

Consider the matrices

$$\mathbf{A}=\begin{bmatrix} 2 & 0 \\ -4 & 6 \end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix} 1 & -7 & 2 \\ 5 & 3 & 0 \end{bmatrix},\qquad \mathbf{C}=\begin{bmatrix} 4 & 9 \\ -3 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix} -2 & 1 & 8 \\ 3 & 0 & 2 \\ 4 & -6 & 3 \end{bmatrix},\qquad \mathbf{E}=\begin{bmatrix} 0 & 3 & 0 \\ -5 & 1 & 1 \\ 7 & 6 & 2 \end{bmatrix},\qquad \mathbf{F}=\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}$$

and evaluate each of the following expressions.

(a) $\mathbf{A}(\mathbf{B}\mathbf{C})$

(b) $\mathrm{Tr}(4\mathbf{E}^T-\mathbf{D})$

(c) $\mathrm{Tr}(\mathbf{F}\mathbf{F}^T)$

Problem 2.3

Find all values of k, if any, that satisfy the following equation

$$[2\ \ 2\ \ k]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix}=0$$

Problem 2.4

A matrix is said to be an *orthogonal matrix* if its transpose is the same as its inverse. Thus, a matrix $\mathbf{A}$ is orthogonal if

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

or

$$\mathbf{A}^T\mathbf{A} = \mathbf{I}$$

Show that

$$\mathbf{A} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix.

Regular Problems (To be turned in for grading)

Problem 2.1★

A matrix $\mathbf{B}$ is said to be a *Square Root* of a matrix $\mathbf{A}$ if

$$\mathbf{BB} = \mathbf{A}$$

(a) Find *two* square roots of

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Hint: You can begin by noting that since $\mathbf{A}$ is symmetric, then the square root is probably also symmetric as well. So, let

$$\mathbf{B} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and then multiply out the equation $\mathbf{BB} = \mathbf{A}$ and try to solve for a, b, and c by equating the coefficients on each side of the equation.

(b) How many different square roots can you find if

$$\mathbf{A} = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}$$

(c) Do you think that every 2×2 matrix has at least one square root? Explain your reason.

Problem 2.2★

In real arithmetic, we know that

$$(a+b)^2 = a^2 + 2ab + b^2$$

Does this relationship also hold if a and b are matrices, i.e., is it always true that

$$(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$$

Explain why or why not.

Problem 2.3★

Determine whether or not the following matrices are invertible (do not use MATLAB). Note that here you are to *show* whether or not the matrix is invertible. You do not need to find the inverse (if it exists).

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

Problem 2.4★

A square matrix is said to be *idempotent* if $\mathbf{A}^2 = \mathbf{A}$.

(a) Show that if $\mathbf{A}$ is idempotent, then so is $\mathbf{I} - \mathbf{A}$.

(b) Show that if $\mathbf{A}$ is idempotent, then $2\mathbf{A} - \mathbf{I}$ is *invertible* and is its own inverse, i.e.,

$$(2\mathbf{A} - \mathbf{I})^{-1} = 2\mathbf{A} - \mathbf{I}$$

Problem 2.5★

A square matrix $\mathbf{A}$ is said to be *nilpotent* if $\mathbf{A}^k = \mathbf{0}$ for some positive integer k, and the smallest number k for which this is true is said to be the *degree* of the matrix.

(a) Is the following matrix nilpotent? If so, what is its degree?

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 5 & 7 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Repeat part (a) for the following matrix,

$$\mathbf{B} = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

(c) Show that if $\mathbf{A}$ is nilpotent of degree n, then $\mathbf{I} - \mathbf{A}$ is invertible and

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}$$

Hint: First show that for any integer n,

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1}) = \mathbf{I} - \mathbf{A}^n$$

(Note that the scalar form of this formula is the same one that is used to derive the geometric series expansion).