Differentiation of Real- and Vector Valued Functions

CSE 788.14

Han-Wei Shen

Motivation

- To understand a real world phenomenon represented as a function, we often need to know the rate of change in the function value
- Let's focus on real-valued functions of single variable, that is

$$f : R \rightarrow R$$

Rate of Change

- Assume y = f(x), then the rate of change in y can be written as:

$$\frac{\triangle y}{\triangle x} = \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

- If $\triangle x \rightarrow 0$ then $\frac{\triangle y}{\triangle x}$ is the derivative of f, i.e.

$$f'(x) = \lim_{\triangle x \rightarrow 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

- Remember, f'(x) is only to approximate $\frac{\triangle y}{\triangle x}$

Rate of Change (cont'd)

- We can write the approximation as:

$$\frac{\triangle y}{\triangle x} = f'(x) + E(\triangle x)$$

and when $\triangle x \rightarrow 0, E(\triangle x) \rightarrow 0$

- We can write the infinitesimal $\triangle x, \triangle y$ as:

$\triangle x = dx$ (independent variable)

$\triangle y = dy$ (dependent variable)

(when $\triangle x \rightarrow 0$)

- And

$$dy = f'(x)dx$$

Definition of *limit*

- Suppose I is an open interval that contains c, the function f has a limit L at c if

  for each ε, there exists a $\delta > 0$, such that

  $|f(x)-L| < \varepsilon$ for all $x \in I$ that satisfy

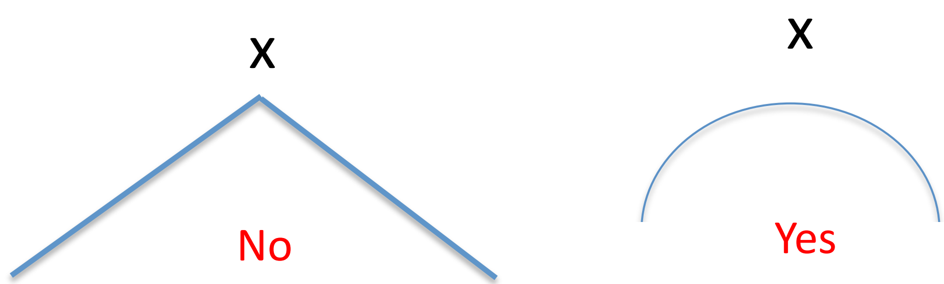
  $|x-c| < \delta$, we write

$$\lim_{x \to c} f(x) = L$$

Differentiability

- If f is defined at x, we say f is *differentiable* iff f'(x) exists at x, i.e

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$ exists.



- Twice differentialbe - f''(x) exists

Continuity

- The function f is *continuous* at p if
  for every $\varepsilon > 0$, there exists a $\delta > 0$ such that
  $|f(x) - f(p)| < \varepsilon$ for all x that satisfy
  $|x - p| < \delta$
- Theorem: If f is *differentiable* at x, then f is *continuous* at x (is the other way true?)

Higher Order Derivatives

$$f, f', f'', f^{(3)}, f^{(4)}, ..., f^{(n)}$$

- Every $f^{(n)}$ is the derivative of $f^{(n-1)}$
- In order for $f^{(n)}$ to exist at x, $f^{(n-1)}$ must exist in the neighborhood of x, and $f^{(n-1)}$ must be differentiable at x, and $f^{(n-2)}$ must be differentiable at x ..., etc.
- C^0 continuous at x means f is continuous at x

  C^1 continuous at x means f' is continuous at x

  C^2 continuous at x means f'' is continuous at x

Taylor Series

- Examples of Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{x!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + ... = \sum_{k=0}^{\infty} x^k$$

Taylor Series

- Taylor series are often used to compute approximation values of complicated functions at specific points

$$e^8 = 1 + 8 + \frac{64}{2} + \frac{512}{6} + \frac{4096}{24} + ...$$

- The above example needs many terms to approximate the function correctly

Taylor Series for f at c

$$f(x) \sim f(c) + f^{'}(c)(x-c) + \frac{f^{''}(c)}{2!}(x-c)^2 + \frac{f^{'''}(c)}{3!}(x-c)^3 + ...$$

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

- A Taylor series converges rapidly near the point of expansion (value at c) and slowly at more remote point

Taylor Theorem

- If a function f possesses continuous derivatives of orders 0,1,2, …, (n+1) in a closed interval I = [a,b], then for any c, and x in I

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

The error term

where

$$E_{n+1} = \frac{f^{n+1}(\varepsilon)}{(n+1)!}(x-c)^{n+1} \quad , \quad \varepsilon$$ lies between C and X

Mean-Value Theorem

- A special case of Taylor's theorem

  If f is a continuous function on the closed interval [a,b] and possesses a derivative at each point in (a,b), then

$$f(b) = f(a) + (b-a)f^{'}(\varepsilon)$$

Taylor's Theorem for f(x+h)

- f possesses derivatives of order 0,1,2,…(n+1) in a closed interval I = [a,b], then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

where

$$E_{n+1} = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{n+1} \qquad \begin{array}{l} , \varepsilon \in (x, x+h), h > 0 \\ , \varepsilon \in (x+h, x), h < 0 \end{array}$$

The Error Term E_{n+1}

- E_{n+1} will converge to zero as quickly as h converges to zero

$$E_{n+1} = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{n+1}$$

- We can write E_{n+1} as

  $E_{n+1} = O(h^{n+1})$ (the big O notation), i.e.,

$$|E_{n+1}| \leq C|h|^{n+1}$$

Taylor Theorem and Error Estimate

- Taylor theorem can be used to estimate errors in a numerical process

$$f(x+h) = f(x) + f^{'}(\varepsilon_1)h = f(x) + O(h)$$

$$f(x+h) = f(x) + f^{'}(x)h + \frac{1}{2!}f^{''}(\varepsilon_2)h^2$$
$$= f(x) + f^{'}(x)h + O(h^2)$$

…

Numerical Differentiation

- Derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Numerical approximation:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- Error:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{f''(x)}{2!}h - \frac{f'''(x)}{3!}h^2 - \cdots$$

Central Differences

$$(1) \quad f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

$$(2) \quad f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \cdots$$

$(1)-(2):$

$$f(x+h) - f(x-h) = 2f'(x)h + \frac{2f'''(x)}{3!}h^3 + \cdots$$

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{f'''(x)}{3!}h^2 - \frac{f^{(5)}(x)}{5!}h^4 - \cdots$$

Second Derivative

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots$$

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \cdots$$

Add the two equations:

$$f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + \frac{2f^{(4)}(x)}{4!}h^4 + \cdots$$

Rearrange:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{2f^{(4)}(x)}{4!}h^2 - \cdots$$

Vector-valued Functions

- Still assume the function has a single variable
- But now we are focused on:

$$f : R \rightarrow R^k$$

- In other words,

$$f(x) = (f_1(x), f_2(x), f_3(x), ..., f_k(x))$$

- Example, an one dimensional vector field, or complex function

$$f(x) = f_1(x) + if_2(x)$$

Vector-valued Functions (cont'd)

- A lot of properties for the real-valued functions still apply

$$f'(x) = \lim_{\triangle x \to 0} |\frac{f(x + \triangle x) - f(x)}{\triangle x}|$$

Norm in R^k

or

$$f'(x) = (f_1'(x), f_2'(x), f_3'(x), ..., f_k'(x))$$

Vector-valued Functions (cont'd)

- Same as real-valued functions:

  if *f* is differentiable at *x*, then *f* is continuous, i.e., each of the function f_i is continuous.

- Also, if f and g are continuous, then

  f+g and f.g are continuous (note, f.g is a dot product)

Vector-valued Functions (cont'd)

- The mean value theorem is changed though.

Suppose f is a continuous mapping of [a,b] into R^k and f is differentiable in (a,b), then there exists x $\in$ (a,b) such that

$$|f(a) - f(b)| \leq (b-a)|f'(x)|$$